Uniqueness in labelings of tree-depth-critical graphs

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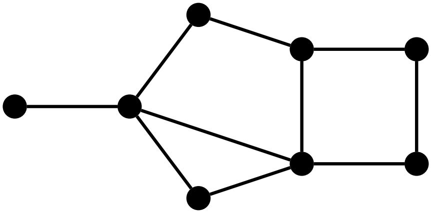
Joint work with John Sinkovic (University of Waterloo)

Tree-depth

(aka (vertex) ranking number, ordered coloring number, ...)



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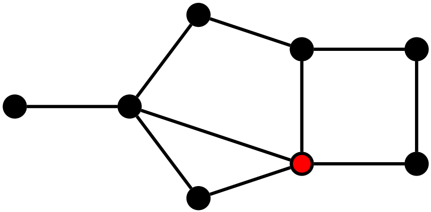

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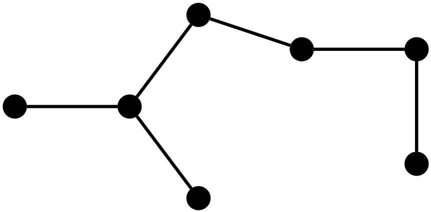

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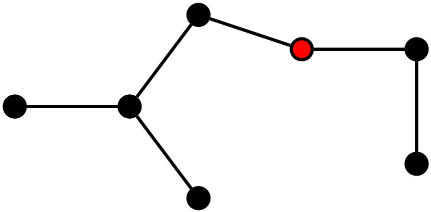

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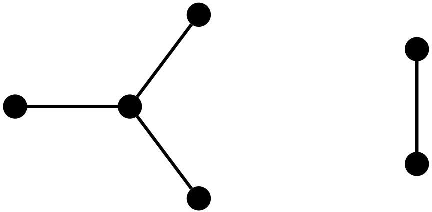

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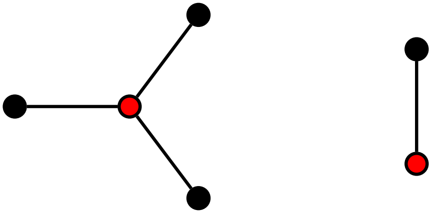

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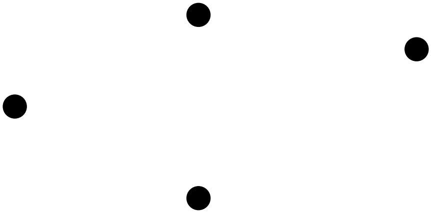

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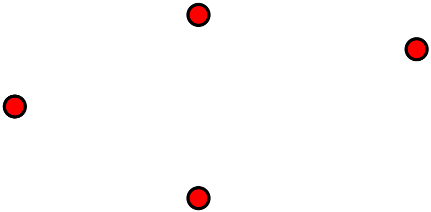

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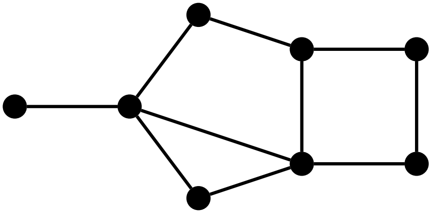

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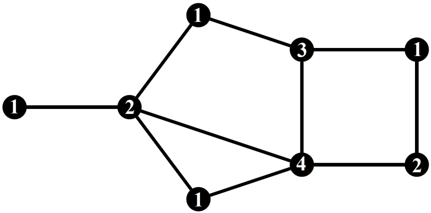

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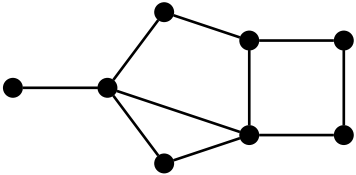
Equivalently, the smallest number of labels needed in a labeling where every path with equal endpoints also has a higher label.

Tree-depth and criticality (with respect to minors)



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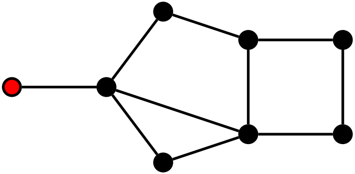

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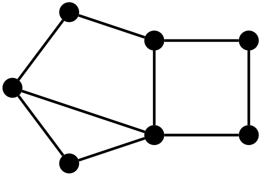

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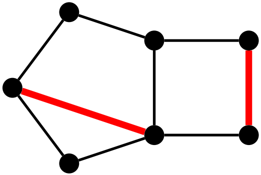

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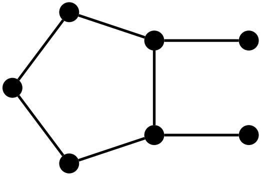

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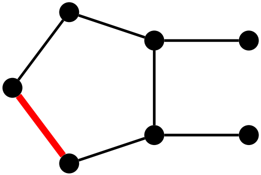

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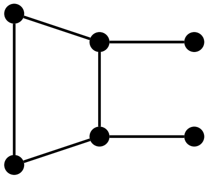

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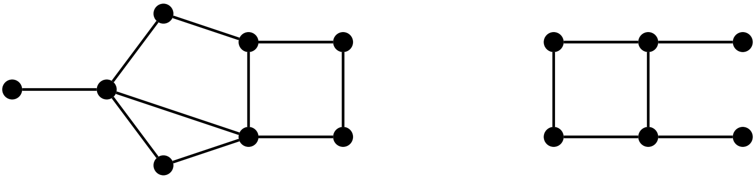
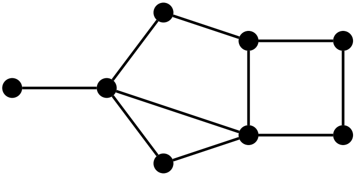
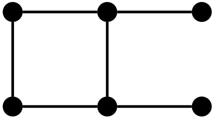


Theorem

If G contains H as a minor, then $\mathrm{td}(G) \geq \mathrm{td}(H)$.

Tree-depth and criticality (with respect to minors)





Theorem

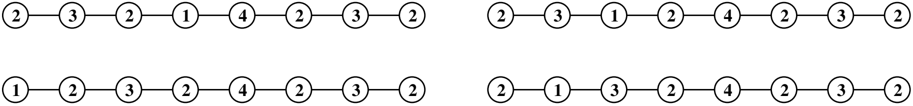
If G contains H as a minor, then $\mathrm{td}(G) \geq \mathrm{td}(H)$.

Idea: A graph is **critical** if every proper minor has lower tree-depth.

Critical minors determine the tree-depth.

New concept: 1-uniqueness

A graph is **1-unique** if for every vertex *v* there exists an optimal ranking where *v* is the only vertex assigned the label 1.



Why 1-uniqueness? It's plausible!

Critical graphs with small tree-depth: Dvořák–Giannopoulou–Thilikos, '09, '12

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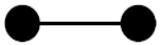
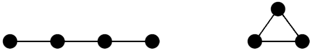
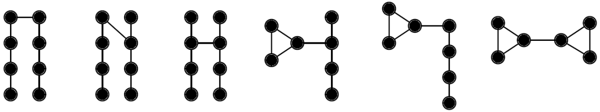
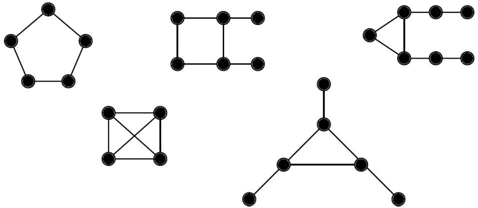
2: 

3: 



4: 

5: 136 trees, plus...

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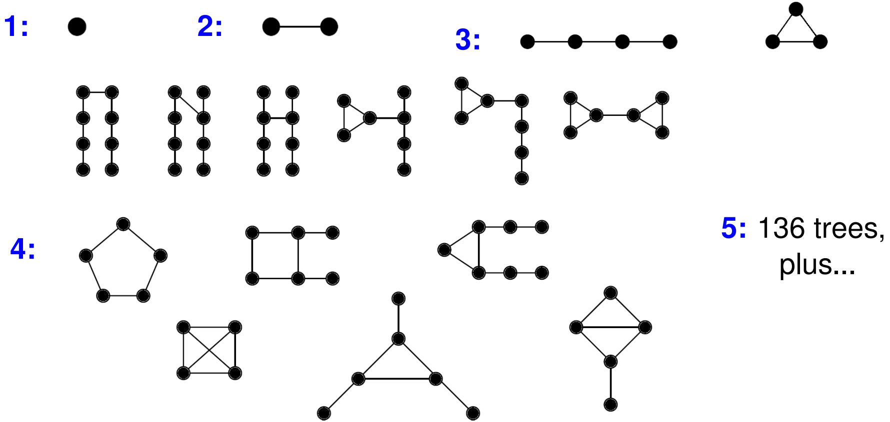


(Ultimately hasty) conjecture: (B, Sinkovic, 2016)

All critical graphs are 1-unique!

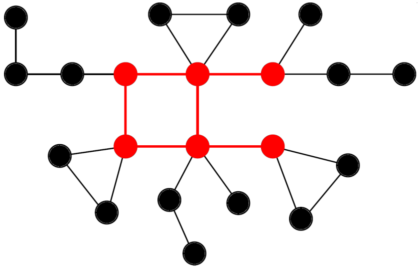
Why 1-uniqueness? It's plausible!

Just like critical graphs, 1-unique graphs are tree-depth-critical with respect to

- **vertex deletions**,
- **edge contractions**, and
- **deletions of edge cuts**.

Why 1-uniqueness? It's useful!

Hang *k*-minor-critical "appendages" off every vertex of an ℓ-minor-critical graph...

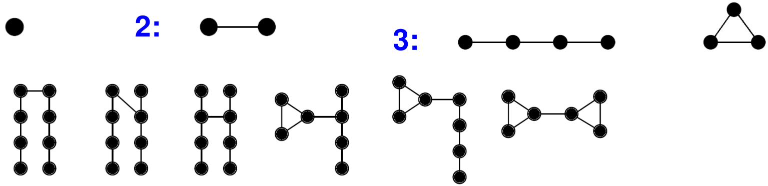


Theorem (B, Sinkovic, 2016)

Graphs constructed in this way are $(k+\ell-1)$*-minor-critical if the appendages are 1-unique.*

Why 1-uniqueness? It's useful!

1: **2:** **3:**



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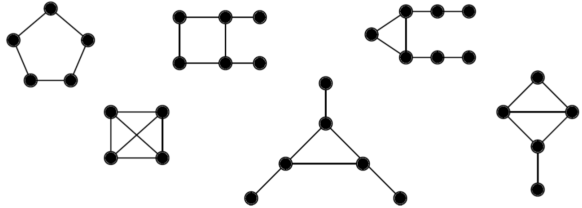

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B, Sinkovic, 2016

Conjecture: If G is k-critical, then $\Delta(G) \leq k - 1$

Proposition: If G is 1-unique and $\text{td}(G) = k$, then $\Delta(G) \leq k - 1$.

Back to that conjecture

(B, Sinkovic, 2016, 2017+)

Is every critical graph 1-unique?

Yes for

- every k-critical graph for $k \in \{1, 2, 3, 4\}$,
- every critical tree,
- every critical cycle,
- every Andrasfai graph,
- ...

Back to that conjecture

(B, Sinkovic, 2016, 2017+)

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Theorem

If G is an n-vertex critical graph ***and*** $\mathrm{td}(G) \geq n - 1$*, then G is 1-unique.*

Resolution

Conjecture

For any k, if G is k-critical, then G is 1-unique.

Known true for

$$k = 1, 2, 3, 4, \qquad\qquad n - 1, n$$

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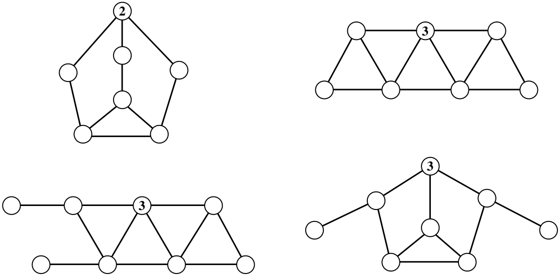
False for

$k = \quad 5, 6, \ldots, n-2$

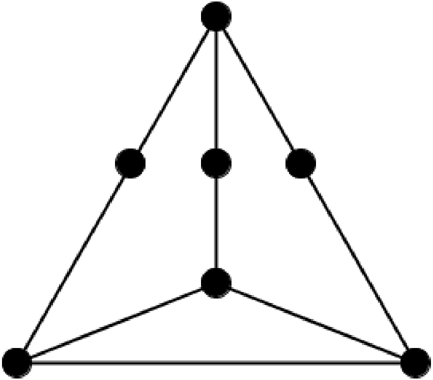
Finding counterexamples



Computer search using SageMath's graph database and functions

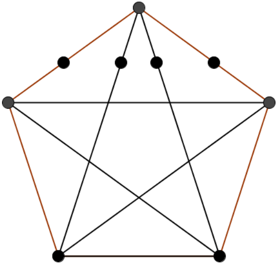
- Iterate through proper colorings of a graph.
- Identify colorings with a unique lowest label; identify 1-unique (or nearly 1-unique) graphs.
- Use 1-uniqueness to produce candidates for criticality testing.

Counterexample



A 7-vertex critical graph G with $\text{td}(G) = 5 = n(G) - 2$ that is not 1-unique!

Counterexamples



For $t \geq 2$, form H_t by subdividing all edges incident with a vertex of K_{t+2}.

Here, $\text{td}(H_t) = n(H_t) - t$.

The graph H_t is critical, but in **no** optimal ranking can the vertex at the subdivided edges' center receive the unique 1.

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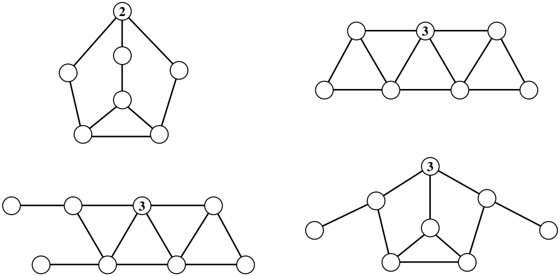
Remaining questions



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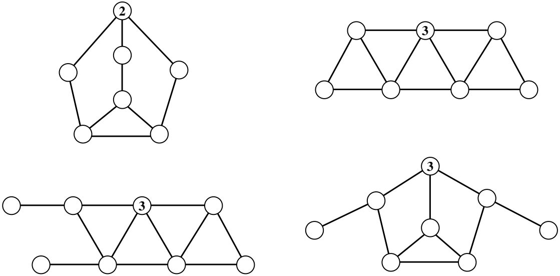

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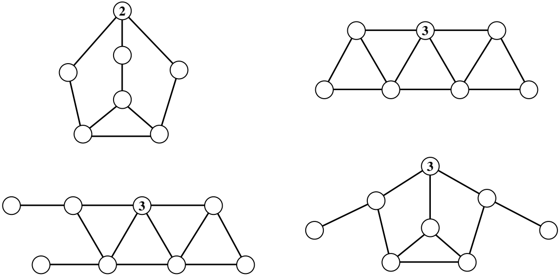

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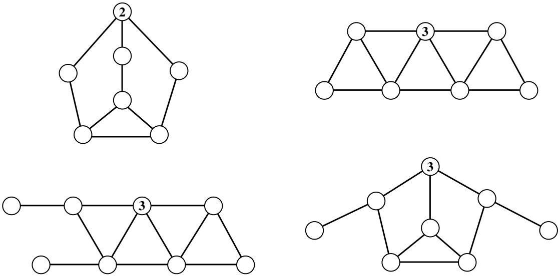
- What fraction of critical graphs are 1-unique?
- How else, instead, can we prove/disprove that critical graphs satisfy $\Delta(G) \leq \text{td}(G) - 1$?

Remaining questions



Thank you!