Degree Sequences, Forced Adjacency Relationships, and Weakly Threshold Graphs

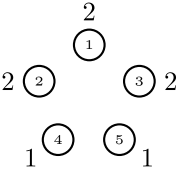
Michael D. Barrus

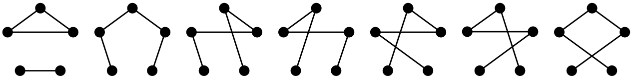
Department of Mathematics, University of Rhode Island

Discrete Math Seminar
November 20, 2015

Realizations and Properties

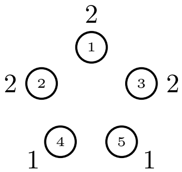
$(2, 2, 2, 1, 1)$

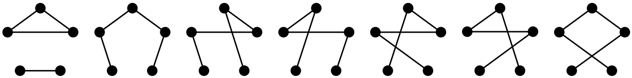




Realizations and Properties

$(2, 2, 2, 1, 1)$

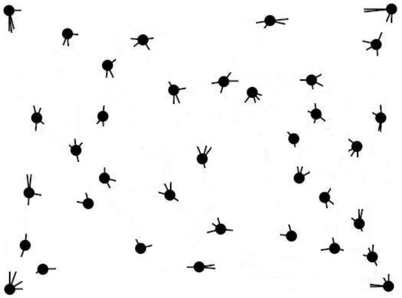




Given a graph property $\mathcal{P}$, a degree sequence *d* is

- **potentially $\mathcal{P}$-graphic** if at least one realization of *d* has property $\mathcal{P}$.
- **forcibly $\mathcal{P}$-graphic** if **every** realization of *d* has property $\mathcal{P}$.

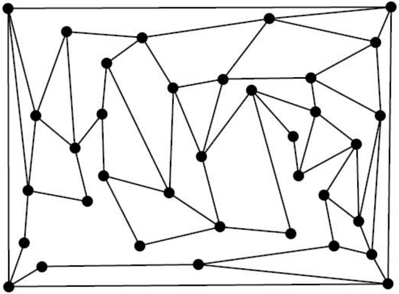
Forcible adjacency relationships



$$d(G) = (4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,\\ 4,4,4,4,3,3,3,3,3,3,3,3,2,2,2,2,2,2)$$

Forcible adjacency relationships

$\mathcal{P}_{ij}$: *ij* is an edge (non-edge)

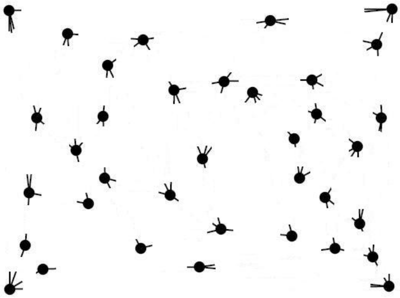


$$d(G) = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2)$$

Are there any forcible edges/non-edges?

Forcible adjacency relationships

$\mathcal{P}_{ij}$: *ij* is an edge (non-edge)

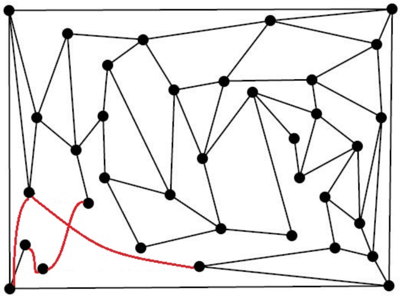


$$d(G) = (4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,\\ 4,4,4,4,3,3,3,3,3,3,3,3,2,2,2,2,2,2)$$

Are there any forcible edges/non-edges?

Forcible adjacency relationships

$\mathcal{P}_{ij}$: ij is an edge (non-edge)

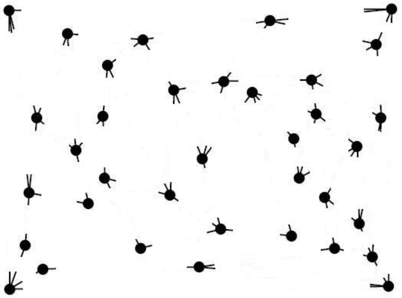


$$d(G) = (4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,\\ 4,4,4,4,3,3,3,3,3,3,3,3,2,2,2,2,2,2)$$

Are there any forcible edges/non-edges?

Forcible adjacency relationships

$\mathcal{P}_{ij}$: *ij* is an edge (non-edge)

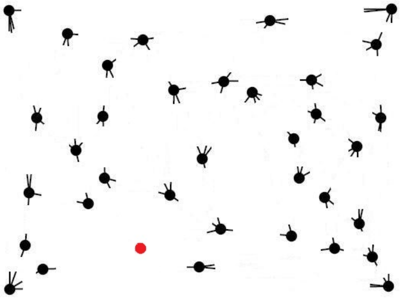


$$d(G) = (4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,\\ 4,4,4,4,3,3,3,3,3,3,3,3,2,2,2,2,2,2)$$

Are there any forcible edges/non-edges?

Forcible adjacency relationships

$\mathcal{P}_{ij}$: *ij* is an edge (non-edge)

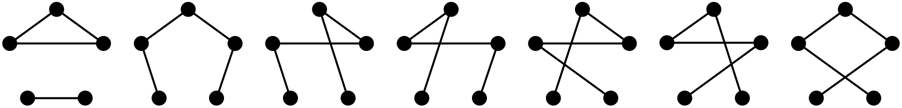


$$d(G) = (4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,\\ 4,4,4,4,3,3,3,3,3,3,3,3,2,2,2,2,2,\mathbf{0})$$

Are there any forcible edges/non-edges?

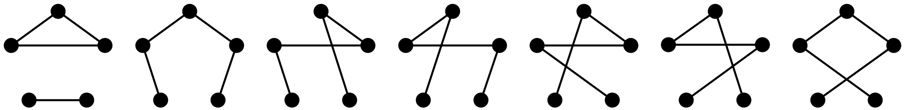
Forcible adjacency relationships: Envelope graphs

$d = (2, 2, 2, 1, 1)$



Forcible adjacency relationships: Envelope graphs

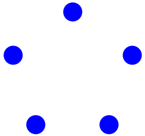
$d = (2, 2, 2, 1, 1)$



Intersection envelope graph I_d

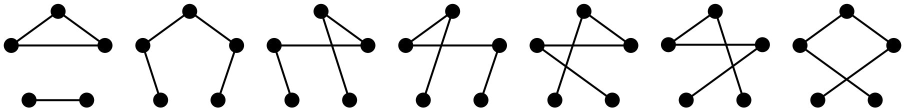
$$E(I_d) = \bigcap_{d(G)=d} E(G)$$

I_d:



Forcible adjacency relationships: Envelope graphs

$d = (2, 2, 2, 1, 1)$



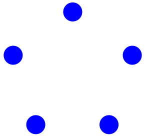
Intersection envelope graph I_d

$$E(I_d) = \bigcap_{d(G)=d} E(G)$$

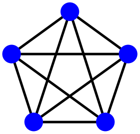
Union envelope graph U_d

$$E(U_d) = \bigcup_{d(G)=d} E(G)$$

I_d:



U_d:



A key graph class: Threshold graphs

Chvátal–Hammer, 1973; many others

Many equivalent characterizations...

threshold sequence: a degree sequence having exactly one (labeled) realization.

threshold graph: a realization of a threshold sequence.

A key graph class: Threshold graphs

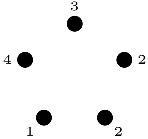
Chvátal–Hammer, 1973; many others

Many equivalent characterizations...

threshold sequence: a degree sequence having exactly one (labeled) realization.

threshold graph: a realization of a threshold sequence.

$$d = (4, 3, 2, 2, 1)$$



A key graph class: Threshold graphs

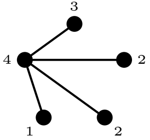
Chvátal–Hammer, 1973; many others

Many equivalent characterizations...

threshold sequence: a degree sequence having exactly one (labeled) realization.

threshold graph: a realization of a threshold sequence.

$$d = (4, 3, 2, 2, 1)$$



A key graph class: Threshold graphs

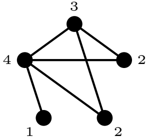
Chvátal–Hammer, 1973; many others

Many equivalent characterizations...

threshold sequence: a degree sequence having exactly one (labeled) realization.

threshold graph: a realization of a threshold sequence.

$$d = (4, 3, 2, 2, 1)$$



A key graph class: Threshold graphs

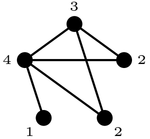
Chvátal–Hammer, 1973; many others

Many equivalent characterizations...

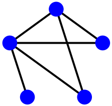
threshold sequence: a degree sequence having exactly one (labeled) realization.

threshold graph: a realization of a threshold sequence.

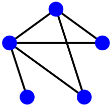
$d = (4, 3, 2, 2, 1)$



I_d:



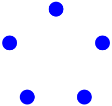
U_d:

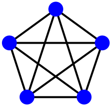


<u>**All**</u> edges and non-edges are forced by the degree sequence.

Forcible adjacency relationships: Envelope graphs

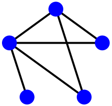
$(2, 2, 2, 1, 1)$

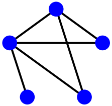
I_d: 

U_d: 

No edges or non-edges are forced by the degree sequence.

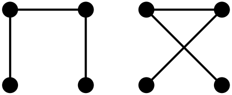
$(4, 3, 2, 2, 1)$

I_d: 

U_d: 

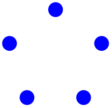
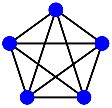
All edges and non-edges are forced by the degree sequence.

$d = (2, 2, 1, 1)$

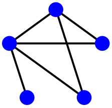
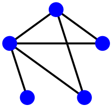


Forcible adjacency relationships: Envelope graphs

$(2, 2, 2, 1, 1)$

I_d:  U_d: 

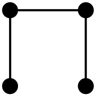
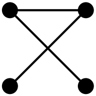
$(4, 3, 2, 2, 1)$

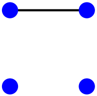
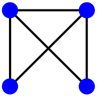
I_d:  U_d: 

No edges or non-edges are forced by the degree sequence.

All edges and non-edges are forced by the degree sequence.

$d = (2, 2, 1, 1)$

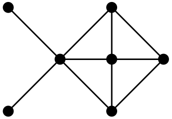
I_d:  U_d: 

Questions

- **How can we recognize forcible adjacency relationships from a degree sequence?**

$$d = (5, 4, 3, 3, 3, 1, 1)$$

- **How can we recognize forcible adjacency relationships from a graph?**



- **What connections are there to interesting graph classes?**

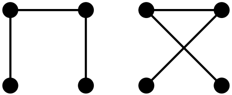
How can we recognize forcible adjacency relationships from a degree sequence?

A beginning

For graphic d and $1 \leq i < j \leq n$, define

$$d^+(i,j) = (d_1, \ldots, d_{i-1}, d_i + 1, d_{i+1}, \ldots, d_{j-1}, d_j + 1, d_{j+1}, \ldots, d_n) \quad \text{and}$$
$$d^-(i,j) = (d_1, \ldots, d_{i-1}, d_i - 1, d_{i+1}, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_n).$$

$d = (2,2,1,1)$ $\qquad$ $d^+(1,3) = (3,2,2,1)$ $\qquad$ $d^+(1,2) = (3,3,1,1)$

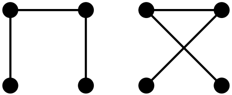


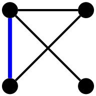
A beginning

For graphic d and $1 \le i < j \le n$, define

$$d^+(i,j) = (d_1, \ldots, d_{i-1}, d_i + 1, d_{i+1}, \ldots, d_{j-1}, d_j + 1, d_{j+1}, \ldots, d_n) \quad \text{and}$$
$$d^-(i,j) = (d_1, \ldots, d_{i-1}, d_i - 1, d_{i+1}, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_n).$$

$d = (2,2,1,1)$ $\qquad$ $d^+(1,3) = (3,2,2,1)$ $\qquad$ $d^+(1,2) = (3,3,1,1)$



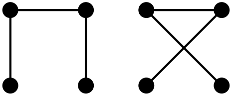


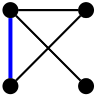
A beginning

For graphic d and $1 \leq i < j \leq n$, define

$$d^+(i,j) = (d_1, \dots, d_{i-1}, d_i + 1, d_{i+1}, \dots, d_{j-1}, d_j + 1, d_{j+1}, \dots, d_n) \quad \text{and}$$
$$d^-(i,j) = (d_1, \dots, d_{i-1}, d_i - 1, d_{i+1}, \dots, d_{j-1}, d_j - 1, d_{j+1}, \dots, d_n).$$

$d = (2, 2, 1, 1)$ $\qquad d^+(1,3) = (3, 2, 2, 1) \qquad d^+(1,2) = (3, 3, 1, 1)$





Lemma

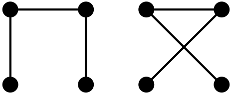
The pair i, j is a forcible $\begin{Bmatrix} \text{edge} \\ \text{non-edge} \end{Bmatrix}$ for d iff $\begin{Bmatrix} d^+(i,j) \\ d^-(i,j) \end{Bmatrix}$ is not graphic.

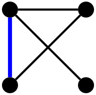
A beginning

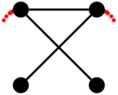
For graphic d and $1 \le i < j \le n$, define

$$d^+(i,j) = (d_1, \ldots, d_{i-1}, d_i + 1, d_{i+1}, \ldots, d_{j-1}, d_j + 1, d_{j+1}, \ldots, d_n) \quad \text{and}$$
$$d^-(i,j) = (d_1, \ldots, d_{i-1}, d_i - 1, d_{i+1}, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_n).$$

$d = (2, 2, 1, 1)$ $\qquad$ $d^+(1,3) = (3, 2, 2, 1)$ $\qquad$ $d^+(1,2) = (3, 3, 1, 1)$







Lemma

The pair i, j is a forcible $\left\{\begin{matrix} \text{edge} \\ \text{non-edge} \end{matrix}\right\}$ for d iff $\left\{\begin{matrix} d^+(i,j) \\ d^-(i,j) \end{matrix}\right\}$ is not graphic.

Erdős–Gallai inequalities

A list $(d_1, \ldots, d_n)$ of nonnegative integers in descending order with even sum is a degree sequence if and only if

$$\underbrace{\sum_{i \le k} d_i}_{\text{LHS}_k(d)} \le \underbrace{k(k-1) + \sum_{i > k} \min\{k, d_i\}}_{\text{RHS}_k(d)}$$

for all $k \le m = \max\{i : d_i \ge i - 1\}$.

Erdős–Gallai inequalities

A list $(d_1, \ldots, d_n)$ of nonnegative integers in descending order with even sum is a degree sequence if and only if

$$\underbrace{\sum_{i \leq k} d_i}_{\mathrm{LHS}_k(d)} \leq \underbrace{k(k-1) + \sum_{i > k} \min\{k, d_i\}}_{\mathrm{RHS}_k(d)}$$

for all $k \leq m = \max\{i : d_i \geq i - 1\}$.

Theorem (Hammer–Ibaraki–Simeone, 1978)

d is a threshold sequence if and only if $\mathrm{LHS}_k(d) = \mathrm{RHS}_k(d)$ for all $k \in \{1, \ldots, m\}$.

Erdős–Gallai differences

A list $(d_1, \ldots, d_n)$ of nonnegative integers in descending order with even sum is a degree sequence if and only if

$$\underbrace{\sum_{i \le k} d_i}_{\text{LHS}_k(d)} \le \underbrace{k(k-1) + \sum_{i>k} \min\{k, d_i\}}_{\text{RHS}_k(d)} \qquad \Delta_k(d) = \text{RHS}_k(d) - \text{LHS}_k(d)$$

for all $k \le m = \max\{i : d_i \ge i-1\}$.

Theorem

Given $1 \le i < j \le n$*,*

$\{i, j\}$ *is a* ***forced edge*** *iff* $\exists k \in \{1, \ldots, n\}$ *such that either*

$\Delta_k(d) = 0$, $i \le k < j$, *and* $k \le d_j$; *or* $\Delta_k(d) \le 1$ *and* $j \le k$.

$\{i, j\}$ *is a* ***forced non-edge*** *iff* $\exists k \in \{1, \ldots, n\}$ *such that either*

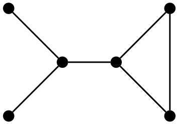
$\Delta_k(d) = 0$, $k < i$, *and* $d_j < k \le d_i$; *or* $\Delta_k(d) \le 1$ *and* $d_i < k < i$.

(7, 6, <u>3</u>, <u>3</u>, <u>3</u>, <u>3</u>, 1, 1, 1)

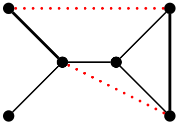
(4, 4, 3, 3, 3, 1)

How can we recognize forcible adjacency relationships from a graph?

A switching result?

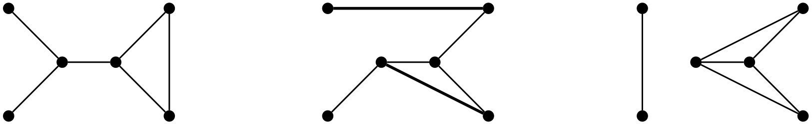


A switching result?

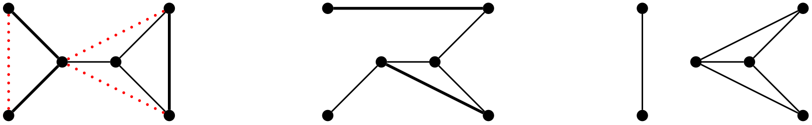


A switching result?

A switching result?



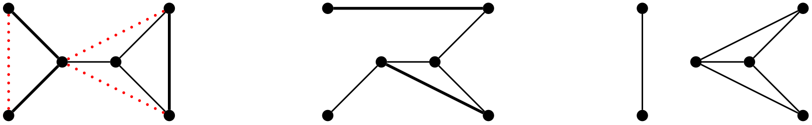
A switching result?



Proposition

The pair $\{i,j\}$ in G is a forcible edge or non-edge for $d(G)$ if and only if $\{i,j\}$ belongs to no alternating circuit in G.

A switching result?



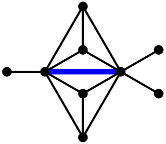
Proposition

The pair $\{i, j\}$ in G is a forcible edge or non-edge for $d(G)$ if and only if $\{i, j\}$ belongs to no alternating circuit in G.

Lots to check...

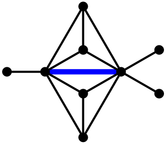
A structural characterization

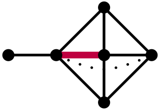
A clique is **demanding** if every vertex outside the clique has as many neighbors as possible in the clique.



A structural characterization

A clique is **demanding** if every vertex outside the clique has as many neighbors as possible in the clique.

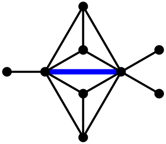


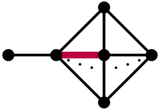


A clique is **weakly demanding** if changing one neighbor of a single vertex outside the clique makes the clique demanding.

A structural characterization

A clique is **demanding** if every vertex outside the clique has as many neighbors as possible in the clique.

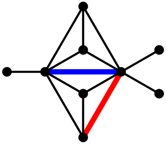




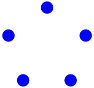
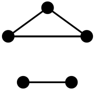
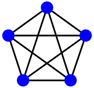
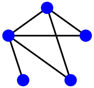
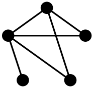
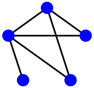
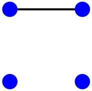
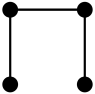
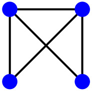
A clique is **weakly demanding** if changing one neighbor of a single vertex outside the clique makes the clique demanding.

Theorem

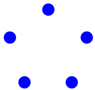
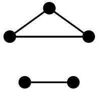
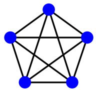
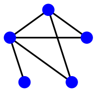
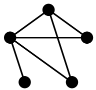
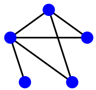
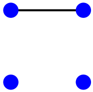
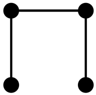
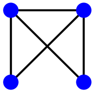
A realization edge is forced for d iff it lies in a demanding or weakly demanding clique or it joins a demanding clique vertex to an external vertex that dominates the clique.



Overall structure of forced relationships

d	I_d	G	U_d
$(2,2,2,1,1)$			
$(4,3,2,2,1)$			
$(2,2,1,1)$			

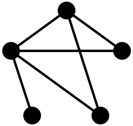
Overall structure of forced relationships

d	I_d	G	U_d
$(2, 2, 2, 1, 1)$			
$(4, 3, 2, 2, 1)$			
$(2, 2, 1, 1)$			

Theorem

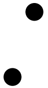
For any degree sequence d, both I_d and U_d are threshold graphs.

Threshold graphs and canonical decomposition

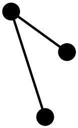


Threshold graphs and canonical decomposition

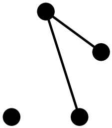
Threshold graphs and canonical decomposition



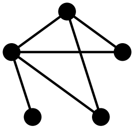
Threshold graphs and canonical decomposition



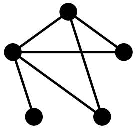
Threshold graphs and canonical decomposition

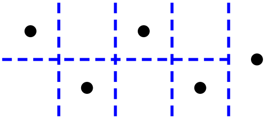


Threshold graphs and canonical decomposition

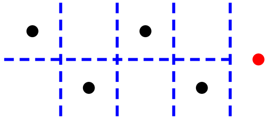


Threshold graphs and canonical decomposition

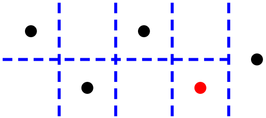




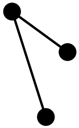
Threshold graphs and canonical decomposition

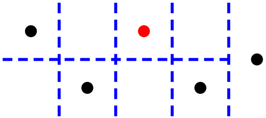


Threshold graphs and canonical decomposition

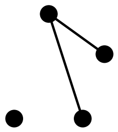


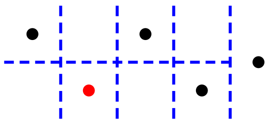
Threshold graphs and canonical decomposition



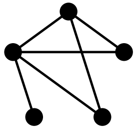


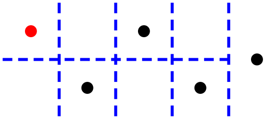
Threshold graphs and canonical decomposition



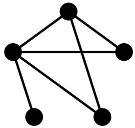


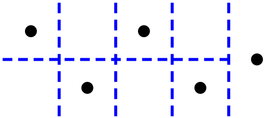
Threshold graphs and canonical decomposition



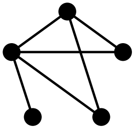


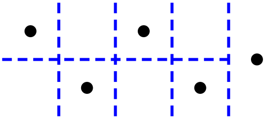
Threshold graphs and canonical decomposition

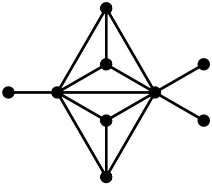


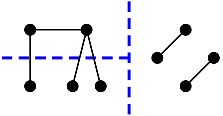


Threshold graphs and canonical decomposition

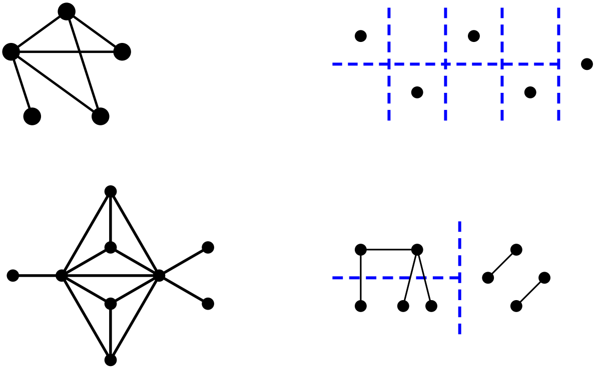






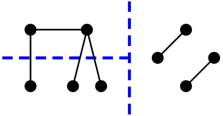


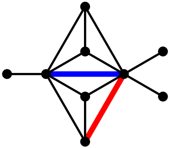
Threshold graphs and canonical decomposition



Canonical decomposition **[Tyshkevich et al., 1980's, 2000]**: Indecomposable split components hooked to each other and an indecomposable "core" following the rightwards dominating/isolated rule; every graph has a unique decomposition, up to isomorphism of canonical components.

Canonical decomposition and forced adjacency relationships





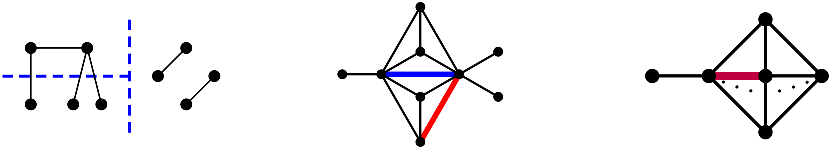
Theorem

For $k \leq m$, the following are equivalent:

- $\mathrm{LHS}_k(d) = \mathrm{RHS}_k(d)$;
- *Vertices* $1, \ldots, k$ *comprise a demanding clique;*
- *Vertices* $1, \ldots, k$ *comprise an initial segment of upper cells in a canonical decomposition.*

Hence all adjacency relationships between vertices in distinct canonical components are forced.

Canonical decomposition and forced adjacency relationships



Theorem

For $k \leq m$, the following are equivalent:

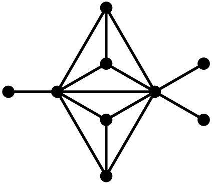
- $\mathrm{LHS}_k(d) = \mathrm{RHS}_k(d)$;
- *Vertices* $1, \ldots, k$ *comprise a demanding clique;*
- *Vertices* $1, \ldots, k$ *comprise an initial segment of upper cells in a canonical decomposition.*

Hence all adjacency relationships between vertices in distinct canonical components are forced.

Overall structure of forced relationships

Theorem

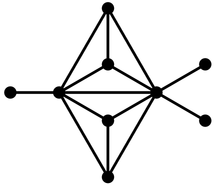
For any degree sequence d, both I_d and U_d are threshold graphs.

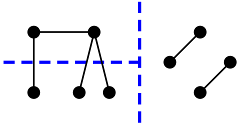


Overall structure of forced relationships

Theorem

For any degree sequence d, both I_d and U_d are threshold graphs.

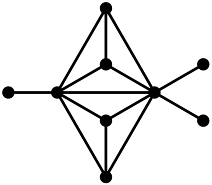


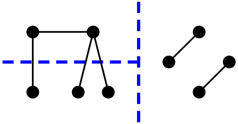


Overall structure of forced relationships

Theorem

For any degree sequence d, both I_d and U_d are threshold graphs.



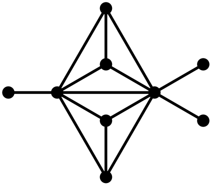


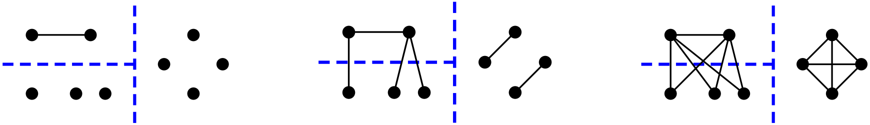
Composing the appropriate envelopes of the individual canonical components, we obtain I_d and U_d.

Overall structure of forced relationships

Theorem

For any degree sequence d, both I_d and U_d are threshold graphs.

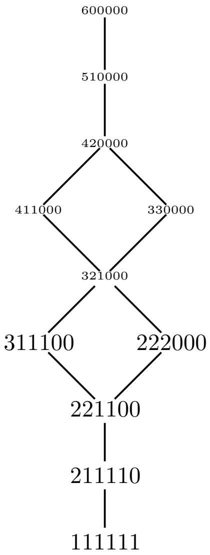




Composing the appropriate envelopes of the individual canonical components, we obtain I_d and U_d.

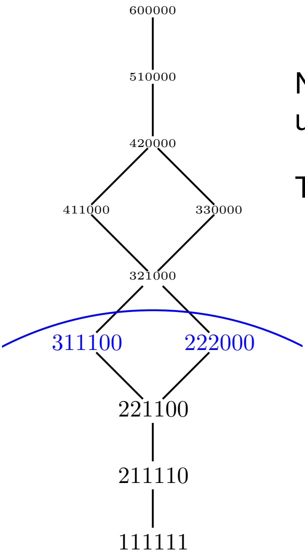
What connections are there to interesting graph classes?

Forced relationships and the dominance order



Nonnegative partitions of $2m$ of a fixed length, under the dominance order

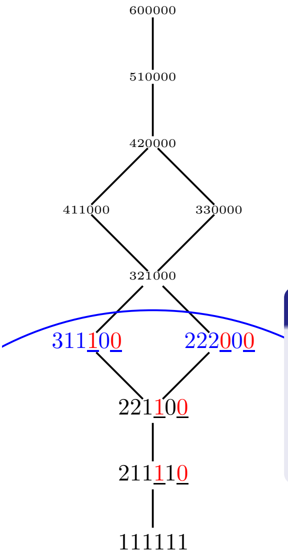
Forced relationships and the dominance order



Nonnegative partitions of $2m$ of a fixed length, under the dominance order

Threshold sequences: maximal graphic elements

Forced relationships and the dominance order



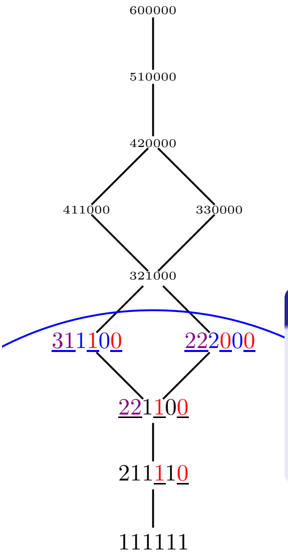
Nonnegative partitions of $2m$ of a fixed length, under the dominance order

Threshold sequences: maximal graphic elements

Theorem

If vertices i and j have a forcible adjacency relationship in realizations of d, then i and j have the same adjacency relationship for all degree sequences that majorize d.

Forced relationships and the dominance order



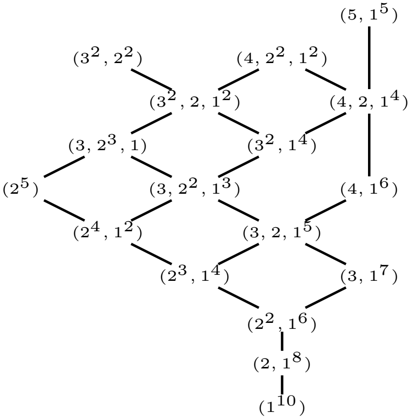
Nonnegative partitions of $2m$ of a fixed length, under the dominance order

Threshold sequences: maximal graphic elements

Theorem

If vertices i and j have a forcible adjacency relationship in realizations of d, then i and j have the same adjacency relationship for all degree sequences that majorize d.

Majorization-closed classes

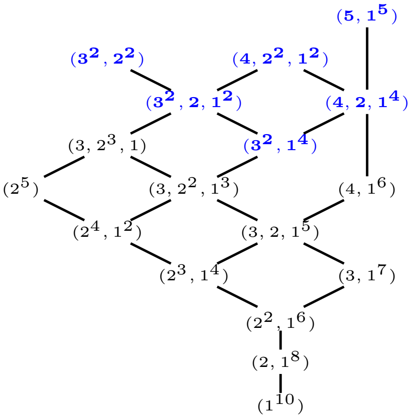


Corollary

Degree sequences for the following classes are "upwards closed" in the poset:

- *[Merris, 2003] Split graphs*
- *Canonically decomposable graphs*

Majorization-closed classes

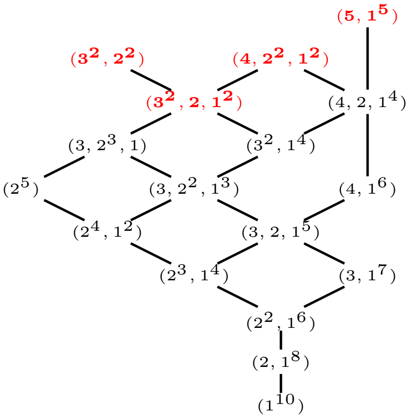


Corollary

Degree sequences for the following classes are "upwards closed" in the poset:

- *[Merris, 2003] Split graphs*
- *Canonically decomposable graphs*

Majorization-closed classes



Corollary

Degree sequences for the following classes are "upwards closed" in the poset:

- *[Merris, 2003] Split graphs*
- *Canonically decomposable graphs*

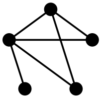
Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

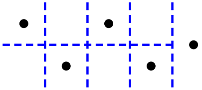
- Equality in the first $m(d)$ Erdős–Gallai inequalities.

  $\sum_{i \le k} d_i = k(k-1) + \sum_{i > k} \min\{k, d_i\}$

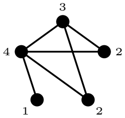
- Iterative construction via dominating/isolated vertices



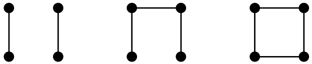
- There are exactly 2^{n-1} threshold graphs on n vertices.



- Unique realization of degree sequence



- $\{2K_2, P_4, C_4\}$-free



- Threshold sequences majorize all other degree sequences

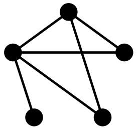
- ...

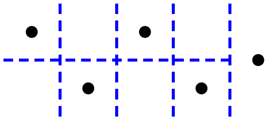
Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

Construction one vertex at a time

Canonical decomposition



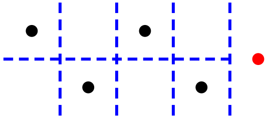


Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

Construction one vertex at a time

Canonical decomposition

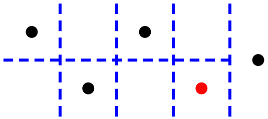


Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

Construction one vertex at a time

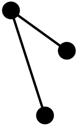
Canonical decomposition



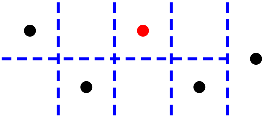
Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

Construction one vertex at a time



Canonical decomposition

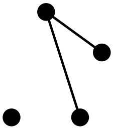


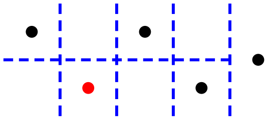
Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

Construction one vertex at a time

Canonical decomposition

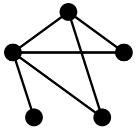




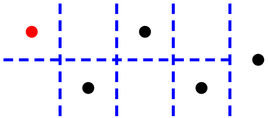
Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

Construction one vertex at a time



Canonical decomposition

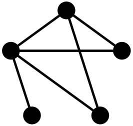


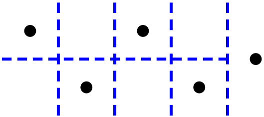
Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

Construction one vertex at a time

Canonical decomposition





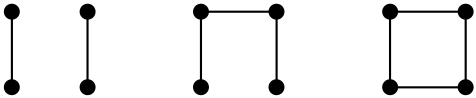
There are exactly 2^{n-1} threshold graphs with n vertices.

Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

Forbidden subgraphs

G is a threshold graph if and only if G has no induced subgraph isomorphic to $2K_2$, P_4, or C_4:

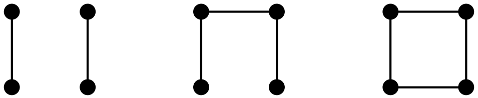


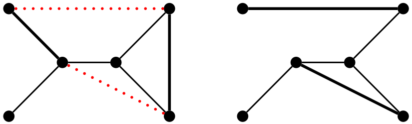
Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

Forbidden subgraphs

G is a threshold graph if and only if G has no induced subgraph isomorphic to $2K_2$, P_4, or C_4:





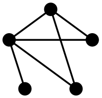
Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

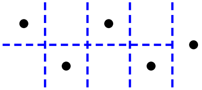
- Equality in the first $m(d)$ Erdős–Gallai inequalities.

  $\sum_{i \leq k} d_i = k(k-1) + \sum_{i>k} \min\{k, d_i\}$

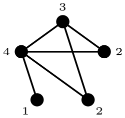
- Iterative construction via dominating/isolated vertices



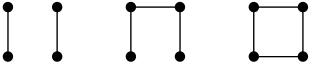
- There are exactly 2^{n-1} threshold graphs on n vertices.



- Unique realization of degree sequence



- $\{2K_2, P_4, C_4\}$-free



- Threshold sequences majorize all other degree sequences

- ...

Erdős–Gallai differences

Which adjacency relationships are forced by d?

$$\underbrace{\sum_{i \le k} d_i}_{\text{LHS}_k(d)} \le \underbrace{k(k-1) + \sum_{i>k} \min\{k, d_i\}}_{\text{RHS}_k(d)}$$

$$\Delta_k(d) = \text{RHS}_k(d) - \text{LHS}_k(d)$$

Erdős–Gallai differences

Which adjacency relationships are forced by d?

$$\underbrace{\sum_{i\le k} d_i}_{\text{LHS}_k(d)} \le \underbrace{k(k-1) + \sum_{i>k} \min\{k, d_i\}}_{\text{RHS}_k(d)}$$

$$\Delta_k(d) = \text{RHS}_k(d) - \text{LHS}_k(d)$$

Theorem

Given $1 \le i < j \le n$,
$\{i,j\}$ *is a* ***forced edge*** *iff* $\exists k \in \{1,\dots,n\}$ *such that either* $\Delta_k(d) = 0$, $i \le k < j$, *and* $k \le d_j$; *or* $\Delta_k(d) \le 1$ *and* $j \le k$.

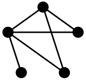
$\{i,j\}$ *is a* ***forced non-edge*** *iff* $\exists k \in \{1,\dots,n\}$ *such that either* $\Delta_k(d) = 0$, $k < i$, *and* $d_j < k \le d_i$; *or* $\Delta_k(d) \le 1$ *and* $d_i < k < i$.

Forcible edges can be determined by examining when $\Delta_k(d) \le 1$.

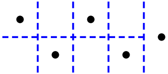
Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

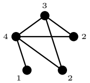
- The first $m(d)$ Erdős–Gallai differences equal 0.
- Iterative construction via dominating/isolated vertices



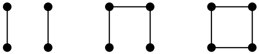
- There are exactly 2^{n-1} threshold graphs on n vertices.



- Unique realization of degree sequence



- $\{2K_2, P_4, C_4\}$-free

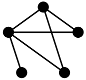


- Threshold sequences majorize all other degree sequences

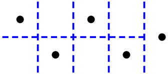
Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

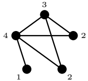
- The first $m(d)$ Erdős–Gallai differences equal 0.
- Iterative construction via dominating/isolated vertices



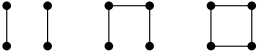
- There are exactly 2^{n-1} threshold graphs on n vertices.



- Unique realization of degree sequence



- $\{2K_2, P_4, C_4\}$-free



- Threshold sequences majorize all other degree sequences

What if $\Delta_k(d) \leq 1$?

Weakly threshold graphs

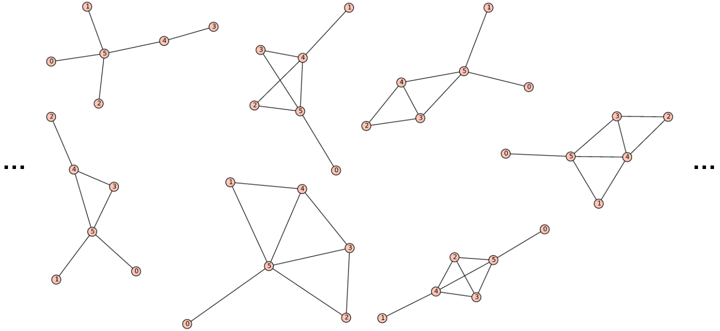
A **weakly threshold sequence** is a **graphic** list $d = (d_1, \ldots, d_n)$ of nonnegative integers in descending order having even sum and satisfying $0 \leq \Delta_k(d) \leq 1$ for all $k \leq \max\{i : d_i \geq i - 1\}$.

A **weakly threshold graph** is a graph having a weakly threshold sequence as its degree sequence.

Weakly threshold graphs

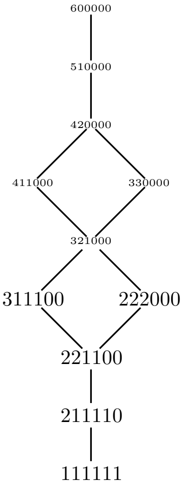
A **weakly threshold sequence** is a **graphic** list $d = (d_1, \ldots, d_n)$ of nonnegative integers in descending order having even sum and satisfying $0 \leq \Delta_k(d) \leq 1$ for all $k \leq \max\{i : d_i \geq i - 1\}$.

A **weakly threshold graph** is a graph having a weakly threshold sequence as its degree sequence.



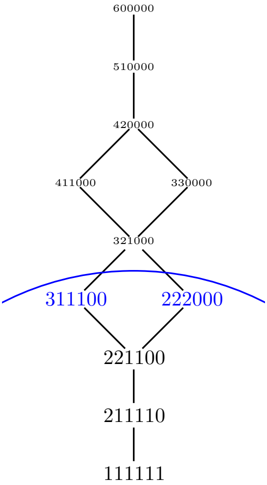
Near the threshold

Threshold sequences majorize all other degree sequences



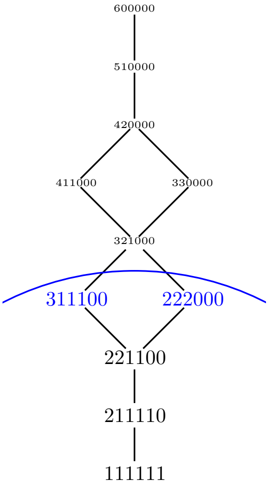
Near the threshold

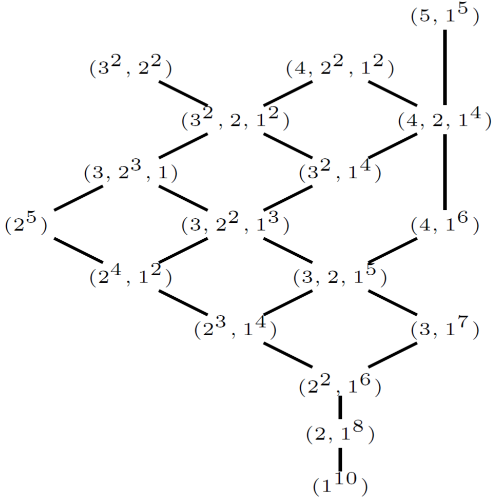
Threshold sequences majorize all other degree sequences



Near the threshold

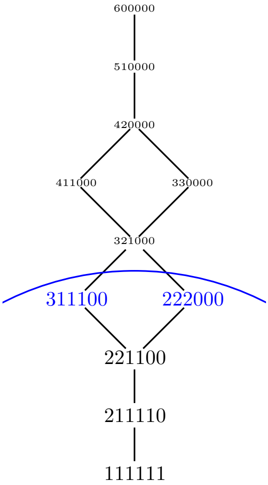
Threshold sequences majorize all other degree sequences

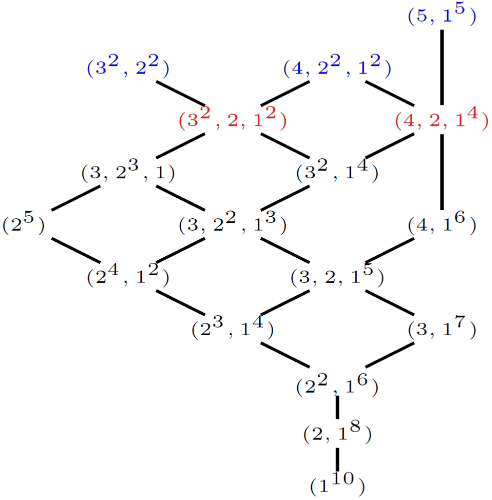




Near the threshold

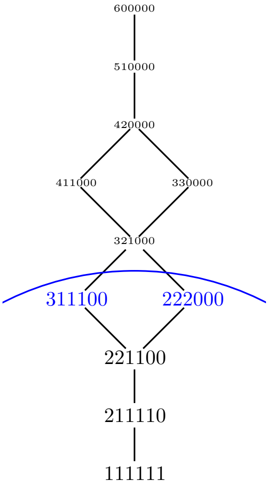
Threshold sequences majorize all other degree sequences

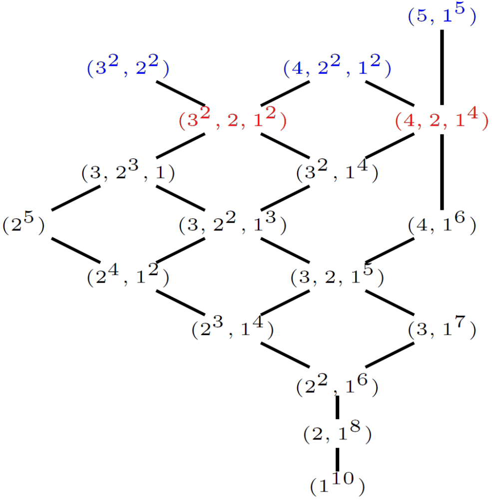




Near the threshold

Threshold sequences majorize all other degree sequences

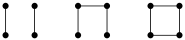




WT sequences are upwards-closed, continue to majorize.

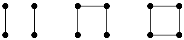
A forbidden subgraph characterization

G is a threshold graph iff G is $\{2K_2, P_4, C_4\}$-free



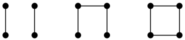
A forbidden subgraph characterization

G is a threshold graph iff G is $\{2K_2, P_4, C_4\}$-free



The class of weakly threshold graphs is **hereditary** (i.e., closed under taking induced subgraphs).

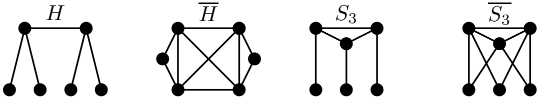
A forbidden subgraph characterization

G is a threshold graph iff G is $\{2K_2, P_4, C_4\}$-free 

The class of weakly threshold graphs is **hereditary** (i.e., closed under taking induced subgraphs).

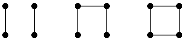
Theorem

A graph G is weakly threshold if and only if it is $\{2K_2, C_4, C_5, H, \overline{H}, S_3, \overline{S_3}\}$-free.



A forbidden subgraph characterization

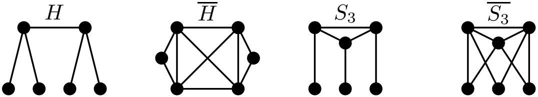
G is a threshold graph iff G is $\{2K_2, P_4, C_4\}$-free



The class of weakly threshold graphs is **hereditary** (i.e., closed under taking induced subgraphs).

Theorem

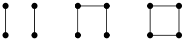
A graph G is weakly threshold if and only if it is $\{2K_2, C_4, C_5, H, \overline{H}, S_3, \overline{S_3}\}$-free.



The class is closed under complementation.
Weakly threshold graphs are all **split** graphs.

A forbidden subgraph characterization

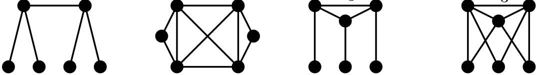
G is a threshold graph iff G is $\{2K_2, P_4, C_4\}$-free



The class of weakly threshold graphs is **hereditary** (i.e., closed under taking induced subgraphs).

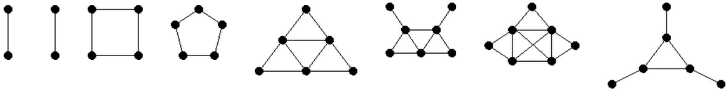
Theorem

A graph G is weakly threshold if and only if it is $\{2K_2, C_4, C_5, H, \overline{H}, S_3, \overline{S_3}\}$-free.



They form a large subclass of **interval** $\cap$ **co-interval.**

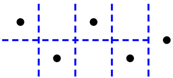
(This class's forbidden induced subgraphs:)



Structural characterization

Threshold iff constructed from K_1 via dominating/ isolated vertices.

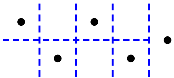
Exactly 2^{n-1} threshold graphs on n vertices.



Structural characterization

Threshold iff constructed from K_1 via dominating/ isolated vertices.

Exactly 2^{n-1} threshold graphs on n vertices.



Theorem

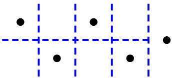
A graph is weakly threshold iff it is constructed by canonically composing special graphs, where

a graph is ***special*** *iff it is isomorphic to K_1 or is obtained by starting with P_4 and iteratively adding either* ***weakly dominating*** *or* ***weakly isolated*** *vertices.*

Structural characterization

Threshold iff constructed from K_1 via dominating/ isolated vertices.

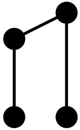
Exactly 2^{n-1} threshold graphs on n vertices.



Theorem

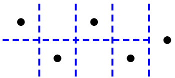
A graph is weakly threshold iff it is constructed by canonically composing special graphs, where

*a graph is **special** iff it is isomorphic to K_1 or is obtained by starting with P_4 and iteratively adding either **weakly dominating** or **weakly isolated** vertices.*



Structural characterization

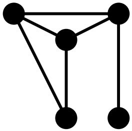
Threshold iff constructed from K_1 via dominating/ isolated vertices.
Exactly 2^{n-1} threshold graphs on n vertices.



Theorem

A graph is weakly threshold iff it is constructed by canonically composing special graphs, where

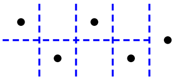
*a graph is **special** iff it is isomorphic to K_1 or is obtained by starting with P_4 and iteratively adding either **weakly dominating** or **weakly isolated** vertices.*



Structural characterization

Threshold iff constructed from K_1 via dominating/ isolated vertices.

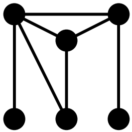
Exactly 2^{n-1} threshold graphs on n vertices.



Theorem

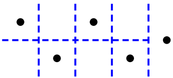
A graph is weakly threshold iff it is constructed by canonically composing special graphs, where

a graph is ***special*** *iff it is isomorphic to K_1 or is obtained by starting with P_4 and iteratively adding either* ***weakly dominating*** *or* ***weakly isolated*** *vertices.*



Structural characterization

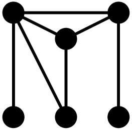
Threshold iff constructed from K_1 via dominating/
isolated vertices.
Exactly 2^{n-1} threshold graphs on n vertices.

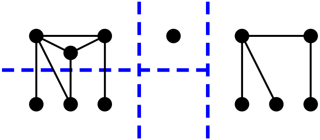


Theorem

A graph is weakly threshold iff it is constructed by canonically composing special graphs, where

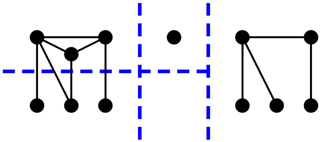
a graph is ***special*** *iff it is isomorphic to K_1 or is obtained by starting with P_4 and iteratively adding either* ***weakly dominating*** *or* ***weakly isolated*** *vertices.*





Iterative construction

Threshold iff constructed from K_1 via dominating/isolated vertices.
Exactly 2^{n-1} threshold graphs on n vertices.



Theorem

A graph is weakly threshold iff it is constructed from K_1 or P_4 by iteratively adding one of

- *a dominating vertex,*
- *an isolated vertex,*
- *a weakly dominating vertex,*
- *a weakly isolated vertex, or*
- *a P_4 with its midpoints dominating all previous vertices.*

Enumeration

Threshold iff constructed from K_1 via dominating/isolated vertices.
Exactly 2^{n-1} threshold graphs on n vertices.

Theorem

A graph is weakly threshold iff it is constructed from K_1 or P_4 by iteratively adding one of

- *a dominating vertex,*
- *an isolated vertex,*
- *a weakly dominating vertex,*
- *a weakly isolated vertex, or*
- *a P_4 with its midpoints dominating all previous vertices.*

Enumeration

Threshold iff constructed from K_1 via dominating/isolated vertices.
Exactly 2^{n-1} threshold graphs on n vertices.

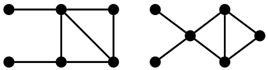
Theorem

A graph is weakly threshold iff it is constructed from K_1 or P_4 by iteratively adding one of

- *a dominating vertex,*
- *an isolated vertex,*
- *a weakly dominating vertex,*
- *a weakly isolated vertex, or*
- *a P_4 with its midpoints dominating all previous vertices.*

Subtleties in direct counting.

Difference between counting degree sequences / isomorphism classes.



Enumeration

Exactly 2^{n-1} threshold graphs on n vertices.

a_n = number of weakly threshold **sequences** of length n

$$(1,)1, 2, 4, 9, 21, 50, 120, 289, 697, 1682, 4060, \ldots$$

Enumeration

Exactly 2^{n-1} threshold graphs on n vertices.

a_n = number of weakly threshold **sequences** of length n

$$(1,)1, 2, 4, 9, 21, 50, 120, 289, 697, 1682, 4060, \ldots$$

OEIS.org sequences A024537, A171842

Theorem

$\{a_n\}_n$ satisfies the following recurrences:

- *For all $n \geq 4$,* $a_n = 2a_{n-1} + \sum_{k=0}^{n-4} 2^k a_{n-4-k}$.
- *For all $n \geq 3$,* $a_n = 3a_{n-1} - a_{n-2} - a_{n-3}$.
- *For all $n \geq 4$,* $a_n = 4a_{n-1} - 4a_{n-2} + a_{n-4}$.
- *For all $n \geq 2$,* $a_n = 2a_{n-1} + a_{n-2} - 1$.

Enumeration

Exactly 2^{n-1} threshold graphs on n vertices.

a_n = number of weakly threshold **sequences** of length n

$$a_n = \frac{2 + (1+\sqrt{2})^n + (1-\sqrt{2})^n}{4} \approx \frac{1}{4} \cdot 2.4^n$$

Enumeration

Exactly 2^{n-1} threshold graphs on n vertices.

a_n = number of weakly threshold **sequences** of length n

$$a_n = \frac{2 + (1+\sqrt{2})^n + (1-\sqrt{2})^n}{4} \approx \frac{1}{4} \cdot 2.4^n$$

From OEIS.org:

- Binomial transform of $1, 0, 1, 0, 2, 0, 4, 0, 8, 0, 16, \ldots$
- Number of nonisomorphic n-element interval orders with no 3-element antichain.
- Top left entry of the nth power of $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ or of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
- Number of $(1, s_1, \ldots, s_{n-1}, 1)$ such that $s_i \in \{1, 2, 3\}$ and $|s_i - s_{i-1}| \leq 1$.
- Partial sums of the Pell numbers prefaced with a 1.
- The number of ways to write an $(n-1)$-bit binary sequence and then give runs of ones weakly incrementing labels starting with 1, e.g., 0011010011022203003330044040055555.
- Lower bound of the order of the set of equivalent resistances of $(n-1)$ equal resistors combined in series and in parallel.

Properties of weakly threshold graphs

- The first $m(d)$ Erdős–Gallai differences equal 0 or 1.
- Iterative construction via (weakly) dominating vertices/(weakly) isolated vertices/half-dominating P_4s.
- There are exactly

$$\frac{2+(1+\sqrt{2})^n+(1-\sqrt{2})^n}{4}$$

weakly threshold sequences of length n.

- Constrained realizations of degree sequences
- $\{2K_2, C_4, C_5, H, \overline{H}, S_3, \overline{S_3}\}$-free
- Weakly threshold sequences at the top of the majorization poset
- ?