Erdős–Gallai near-equalities and the graphs that exhibit them

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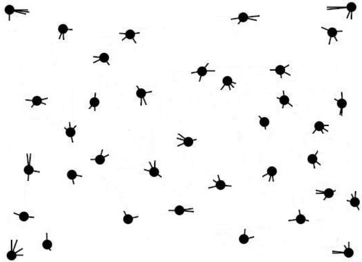
Degree sequences(?)

$$d(G) = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2)$$

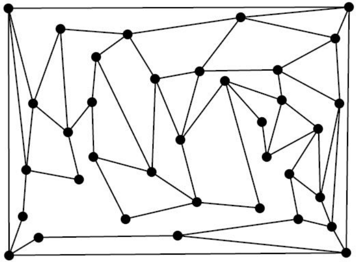
Degree sequences(?)

$$d(G) = (4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,3,3,3,3,3,3,3,3,2,2,2,2,2,2)$$



Degree sequences(?)

$$d(G) = (4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,\\ 4,4,4,4,3,3,3,3,3,3,3,3,2,2,2,2,2,2)$$



Degree sequences(?)

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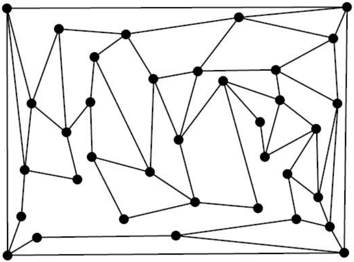
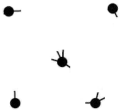
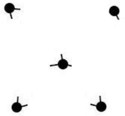


(4,3,1,1,1)



(3,2,2,2,1)



A key criterion

The Erdős–Gallai inequalities (1960)

A list $(d_1, \ldots, d_n)$ of nonnegative integers in descending order with even sum is a degree sequence if and only if

$$\sum_{i \leq k} d_i \leq k(k-1) + \sum_{i>k} \min\{k, d_i\}$$

for all k.

(4,3,1,1,1) (3,2,2,2,1)

A key criterion

The Erdős–Gallai inequalities (1960) See also Hammer–Ibaraki–Simeone (1978)

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for all $k \leq \max\{i : d_i \geq i-1\}$.

(4,3,1,1,1) (3,2,2,2,1)

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(4,3,1,1,1) (3,2,2,2,1)

$4 < 1 \cdot 0 + 1 + 1 + 1 + 1$

$7 > 2 \cdot 1 + 1 + 1 + 1$

A key criterion

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(4,3,1,1,1)

$$4 < 1 \cdot 0 + 1 + 1 + 1 + 1$$
$$7 > 2 \cdot 1 + 1 + 1 + 1$$

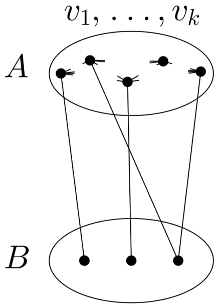
(3,2,2,2,1)

$$3 < 1 \cdot 0 + 1 + 1 + 1 + 1$$
$$5 < 2 \cdot 1 + 2 + 2 + 1$$
$$7 < 3 \cdot 2 + 2 + 1$$

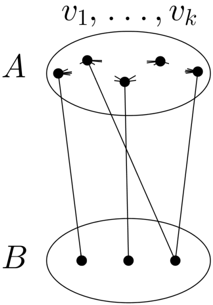
Chapter 1: Spotting Erdős–Gallai *Equalities*

$$\sum_{i \leq k} d_i = k(k-1) + \sum_{i>k} \min\{k, d_i\}$$

Why $\sum_{i \le k} d_i \le k(k-1) + \sum_{i>k} \min\{k, d_i\}$



Why $\sum_{i \le k} d_i \le k(k-1) + \sum_{i>k} \min\{k, d_i\}$



$\sum_{i \le k} d_i - k(k-1)$: Lower bound on number of edges between A and B

$\sum_{i>k} \min\{k, d_i\}$: Upper bound on number of edges between A and B

What if equality holds?

Assuming $k \le \max\{i : d_i \ge i-1\}$

$$\sum_{i \le k} d_i - k(k-1) = \#\text{ edges leaving } A = \sum_{i>k} \min\{k, d_i\}$$

What if equality holds?

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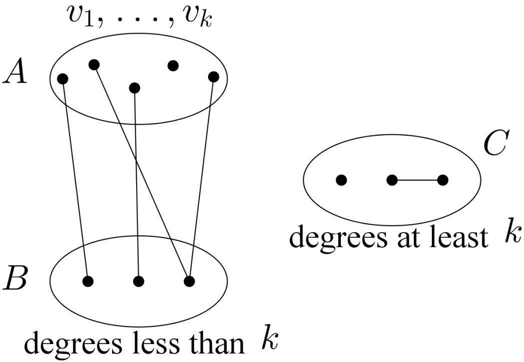

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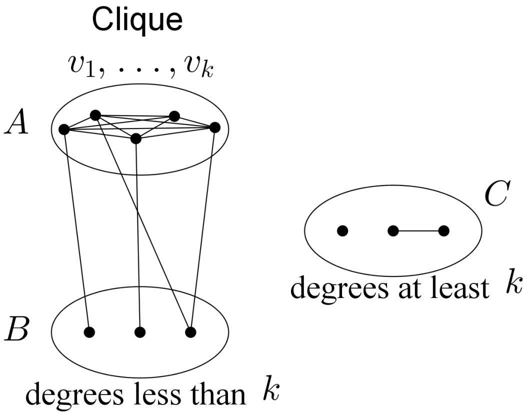

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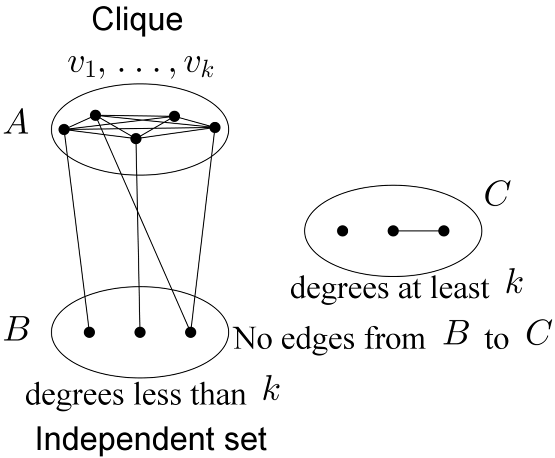

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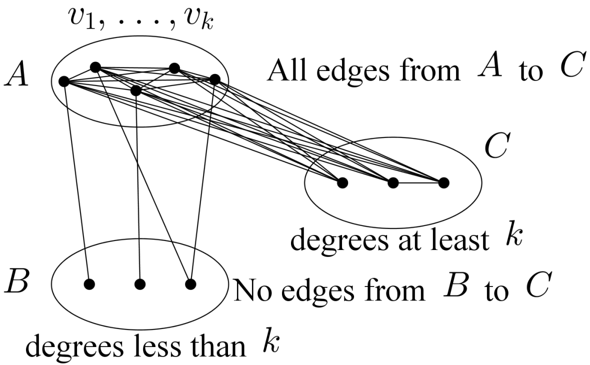
Clique



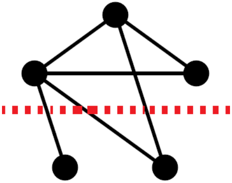
Independent set

First sightings

Split graphs (Hammer–Simeone, 1981)

$$\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i,$$

where $m = \max\{i : d_i \geq i - 1\}$.

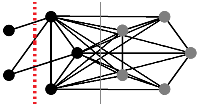


Pseudo-split graphs (Blázsik et al., 1993; Maffray–Preissmann, 1994)

$$\sum_{i=1}^{m} d_i = m(m-1) + 5m + \sum_{i=m+6}^{n} d_i \quad \text{and}$$

$$d_{m+1} = d_{m+2} = d_{m+3} = d_{m+4} = d_{m+5} = m+2,$$

where $m = \max\{i : d_i \geq i - 1\}$.



What if more than one equality holds?

Iterated partitioning

For $d = (9, 9, 7, 7, 6, 6, 6, 3, 3, 1, 1)$,

$$9 + 9 = 2 \cdot 1 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 1 + 1$$

$$9 + 9 + 7 + 7 = 4 \cdot 3 + 4 + 4 + 4 + 3 + 3 + 1 + 1$$

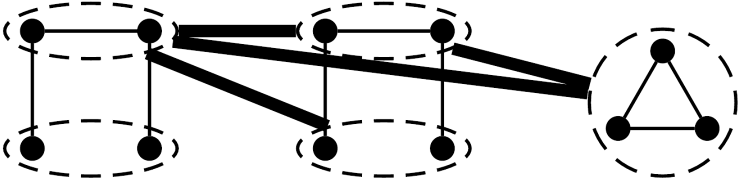
What if more than one equality holds?

Iterated partitioning

For $d = (9, 9, 7, 7, 6, 6, 6, 3, 3, 1, 1)$,

$$9 + 9 = 2 \cdot 1 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 1 + 1$$

$$9 + 9 + 7 + 7 = 4 \cdot 3 + 4 + 4 + 4 + 3 + 3 + 1 + 1$$



Classes with multiple equalities

Matrogenic/matroidal graphs (Tyshkevich, 1984; see also B, 2013)

For each consecutive pair k, k' of indices with EG-equality with $k \geq k + 2$,

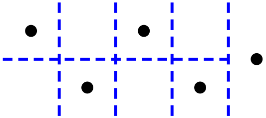
- terms d_i with $k < i \leq k'$ all equal, in $\{d^*_{k'}, d^*_k - \delta_k - 2\}$
- terms d_i with $k < d_i < k'$ all equal, in $\{k + 1, k' - 1\}$

Terms past the last index t of EG-equality, with $d_i > t$, collectively form one of

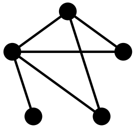
$$((t+1)^{2r}), ((t+2r-2)^{2r}), ((t+2)^5).$$

Threshold graphs

(Hammer–Ibaraki–Simeone, 1978)

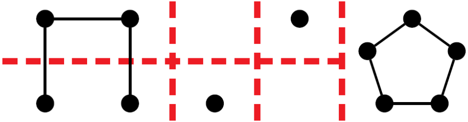


$$\sum_{i=1}^{k} d_i = k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$$



for <u>all</u> $k \leq \max\{i : d_i \geq i - 1\}$.

Tyshkevich's "canonical decomposition" ($\sim$ 1980, 2000)



Theorem

Every graph F can be represented uniquely as a composition

$$F = (G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ F_0$$

of indecomposable components.

(G_i, A_i, B_i): indecomposable splitted graphs

F_0: indecomposable graph

(B, 2013) Partition can be recognized via Erdős–Gallai equalities.

Other graph families with EG-equality connections

via the canonical decomposition

Unigraphs (Tyshkevich–Chernyak, 1978–1979, 2000)

Box-threshold graphs (Tyshkevich–Chernyak, 1985)

Hereditary unigraphs (B, 2013)

Decisive graphs (B, 2014)

Most interesting families with characterizations purely in terms of a degree sequence?

The bottom line

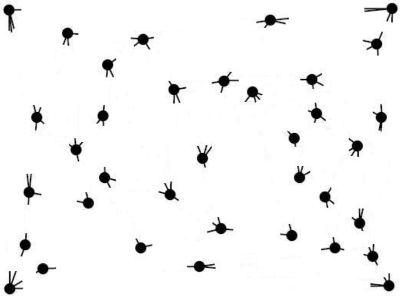
A number of interesting graph classes
have degree sequence characterizations that can be rephrased as
having one or more Erdős–Gallai inequalities hold *with equality*.

Chapter 2: Why are we studying Erdős–Gallai equalities?

$$\sum_{i\leq k} d_i = k(k-1) + \sum_{i>k} \min\{k, d_i\}$$

A question

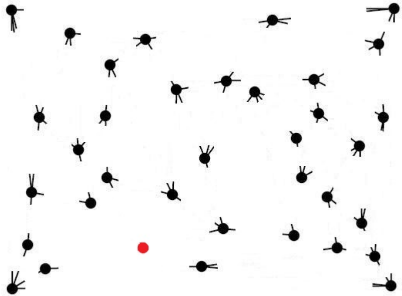
Are there any edges or non-edges *forced* by the degree sequence?



$$d(G) = (4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,\\ 4,4,4,4,3,3,3,3,3,3,3,3,2,2,2,2,2,2)$$

A question

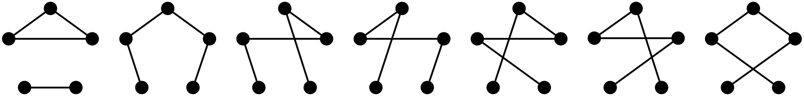
Are there any edges or non-edges *forced* by the degree sequence?



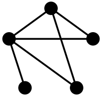
$$d(G) = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, \\ 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, \mathbf{0})$$

Forcible edges and/or non-edges?

$d = (2, 2, 2, 1, 1)$



$d = (4, 3, 2, 2, 1)$



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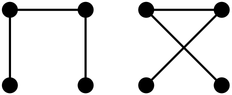

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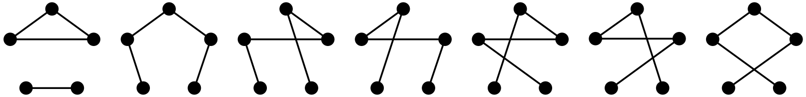

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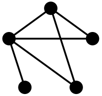
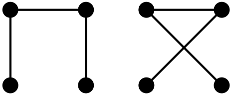


(Hammer–Ibaraki–Simeone, 1978) Threshold graphs are the unique labeled realizations of their degree sequences—every edge, non-edge is forced by the degree sequence

$d = (2, 2, 1, 1)$



Forcible edges and non-edges: results

$$\text{Notation: } \underbrace{\sum_{i \le k} d_i}_{\text{LHS}_k(d)} \le \underbrace{k(k-1) + \sum_{i>k} \min\{k, d_i\}}_{\text{RHS}_k(d)}$$

Theorem (B, 2017+)

Given $1 \le i < j \le n$*,*

$v_i v_j$ *is a **forced edge** iff* $\exists k \in \{1, \ldots, n\}$ *such that*
either $\text{LHS}_k(d) = \text{RHS}_k(d)$, $i \le k < j$, *and* $k \le d_j$,
or $\text{LHS}_k(d) + 1 = \text{RHS}_k(d)$ *and* $j \le k$.

$v_i v_j$ *is a **forced non-edge** iff* $\exists k \in \{1, \ldots, n\}$ *such that*
either $\text{LHS}_k(d) = \text{RHS}_\text{k}(\text{d})$, $k < i$, *and* $d_j < k \le d_i$,
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Forcible edges and non-edges: results

Notation: $\underbrace{\sum_{i \le k} d_i}_{\text{LHS}_k(d)} \le \underbrace{k(k-1) + \sum_{i>k} \min\{k, d_i\}}_{\text{RHS}_k(d)}$

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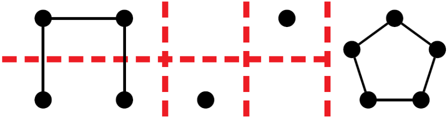


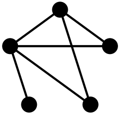
(4, 4, 3, 3, 3, 1)

Blatant foreshadowing

Looking beyond equality,

some interesting things can happen **when** $\mathrm{LHS}_k(d) \approx \mathrm{RHS}_k(d)$**.**

Chapter 3: Pushing the Boundaries on Threshold Graphs



$$\sum_{i \le k} d_i \approx k(k-1) + \sum_{i>k} \min\{k, d_i\}$$

Threshold graphs

Two definitions

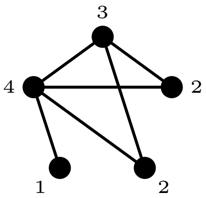
Hammer–Ibaraki–Simeone, 1978:

$$\sum_{i=1}^{k} d_i = k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$$

for <u>all</u> $k \leq \max\{i : d_i \geq i-1\}$.

Chvátal–Hammer, 1973:

Weights on vertices, threshold for adjacency



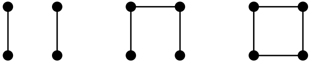
Threshold: 4.5

Other properties of threshold graphs

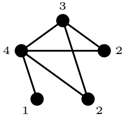
(Chvátal, Hammer, others, 1973+)

- Equality in the first $m(d)$ Erdős–Gallai inequalities.
  $\sum_{i \leq k} d_i = k(k-1) + \sum_{i>k} \min\{k, d_i\}$
- Iterative construction via dominating/isolated vertices

- There are exactly 2^{n-1} threshold graphs on n vertices.
- $\{2K_2, P_4, C_4\}$-free

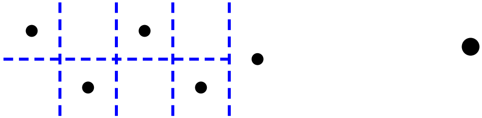


- Unique realization of degree sequence



- Threshold sequences majorize all other degree sequences
- ...

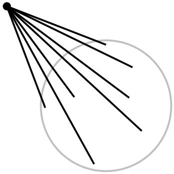
Threshold graph building



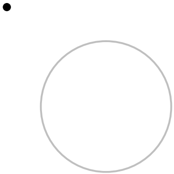
Threshold graph building

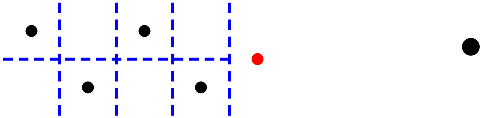
Options for adding

Dominating vertex



Isolated vertex





Threshold graph building

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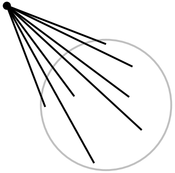
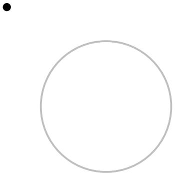
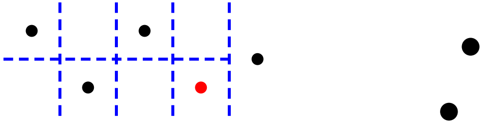


Isolated vertex





Threshold graph building

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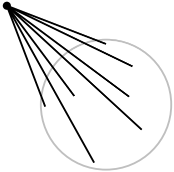
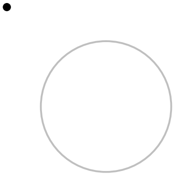
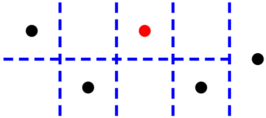
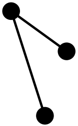


Isolated vertex



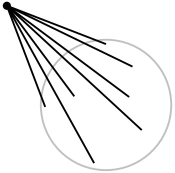




Threshold graph building

Options for adding

Dominating vertex



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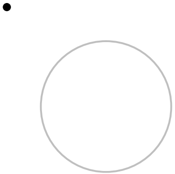

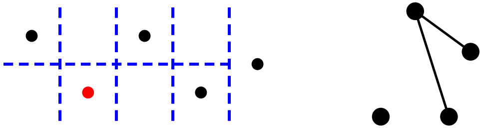

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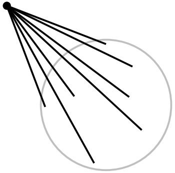

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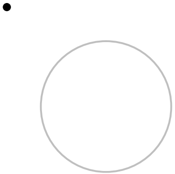

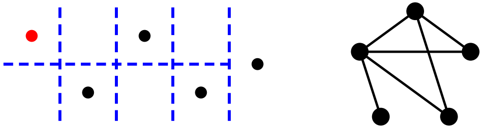

Threshold graph building

Options for adding

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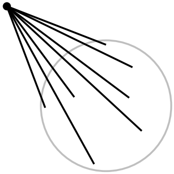
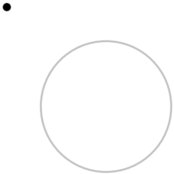
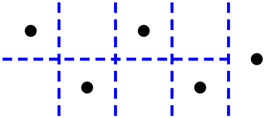
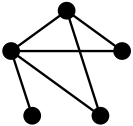


Isolated vertex





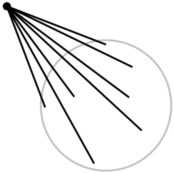


G is a threshold graph if and only if G can be constructed from a single vertex via these operations.

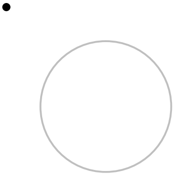
Threshold graph building

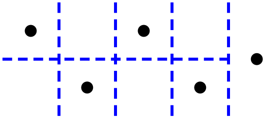
Options for adding

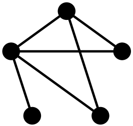
Dominating vertex



Isolated vertex







G is a threshold graph if and only if *G* can be constructed from a single vertex via these operations.

Consequently, up to isomorphism there are exactly 2^{n-1} threshold graphs on n vertices.

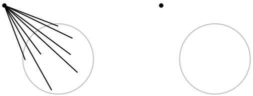
Equality everywhere: properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

- Equality in the first $m(d)$ Erdős–Gallai inequalities.

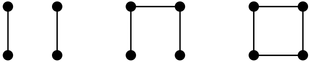
  $\sum_{i\leq k} d_i = k(k-1) + \sum_{i>k} \min\{k, d_i\}$

- Iterative construction via dominating/isolated vertices

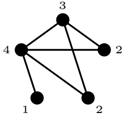


- There are exactly 2^{n-1} threshold graphs on n vertices.

- $\{2K_2, P_4, C_4\}$-free



- Unique realization of degree sequence

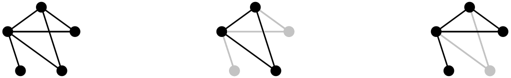


- Threshold sequences majorize all other degree sequences

- ...

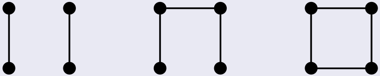
A forbidden subgraph characterization

Induced subgraph: a subgraph obtained by deleting vertices and their incident edges



Theorem (Chvátal–Hammer, 1973)

Any induced subgraph of a threshold graph is a threshold graph. In fact, G is a threshold graph iff G has no induced subgraph isomorphic to one of the following:



(We say that G is $\{2K_2, P_4, C_4\}$-free.)

Threshold sequences and majorization

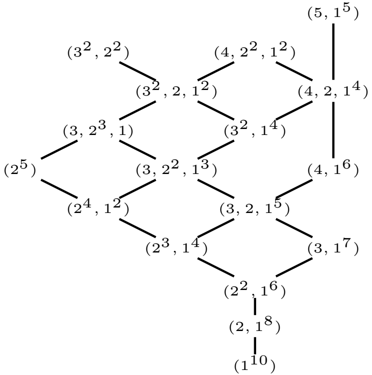
Majorization:

Given $d = (d_1, \ldots, d_n)$ and $e = (e_1, \ldots, e_m)$,

$d \succeq e$ if $\sum d_i = \sum e_i$

and

$$\sum_{i=1}^{k} d_i \geq \sum_{i=1}^{k} e_i \text{ for all } k.$$



Theorem (Ruch–Gutman, 1979; Peled–Srinivasan, 1989)

d is a threshold sequence if and only if d is a maximal element in the poset of all degree sequences with the same sum, ordered by majorization.

Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

- Equality in the first $m(d)$ Erdős–Gallai inequalities.
  $\sum_{i\leq k} d_i = k(k-1) + \sum_{i>k} \min\{k, d_i\}$
- Iterative construction via dominating/isolated vertices

- There are exactly 2^{n-1} threshold graphs on n vertices.
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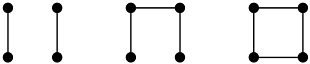

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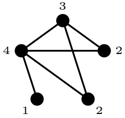

- Threshold sequences majorize all other degree sequences
- ...

What if we required Erdős–Gallai near-equality?

Define a **weakly threshold sequence** to be list $d = (d_1, \ldots, d_n)$ of nonnegative integers in descending order having even sum and satisfying

$$\mathrm{RHS}_k(d) - 1 \leq \mathrm{LHS}_k(d) \leq \mathrm{RHS}_k(d)$$

for all $k \leq \max\{i : d_i \geq i - 1\}$.

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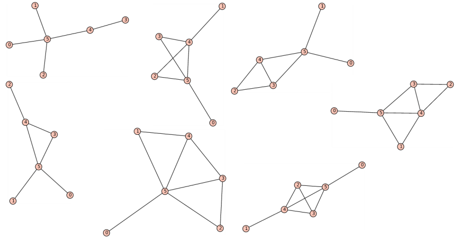
A **weakly threshold graph** will be a graph having a weakly threshold sequence as its degree sequence.

...



...

Weakly threshold sequences and graphs

(B, 2017+)

- Equality or a difference of 1 in each of the first $m(d)$ Erdős–Gallai inequalities.

  $k(k-1) + \sum_{i>k} \min\{k, d_i\} - \sum_{i \leq k} d_i \leq 1$

  Call these **weakly threshold sequences**; call the associated graphs **weakly threshold graphs**.

- Iterative construction?

- How many weakly threshold sequences/graphs on n vertices?

- Forbidden subgraph characterization?

- Unique realizations of degree sequences?

- Majorization result?

- ...?...

Near the threshold (B, 2017+)

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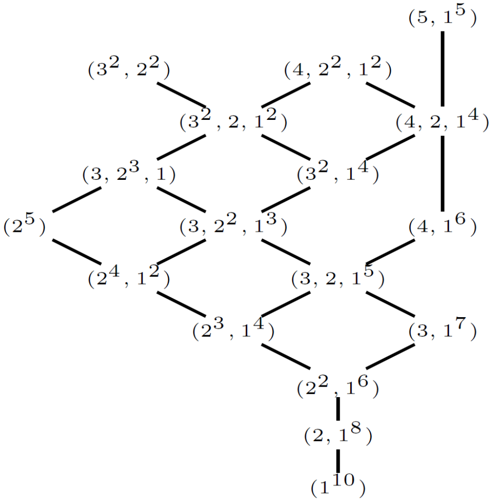

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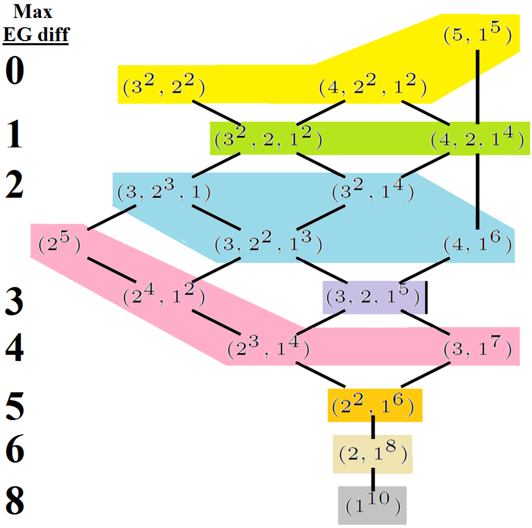

WT sequences (and all diff $\leq b$) are upwards-closed, continue to majorize.

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Iterative construction

Theorem

G is a threshold graph if and only if G can be constructed by beginning with a single vertex and iteratively adding

- *a dominating vertex, or*
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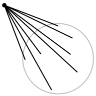
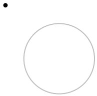
Iterative construction

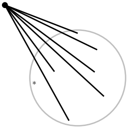


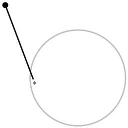


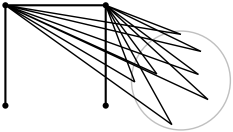
Theorem

*G is a **weakly** threshold graph if and only if G can be constructed by beginning with a single vertex **or** P_4 and iteratively adding*

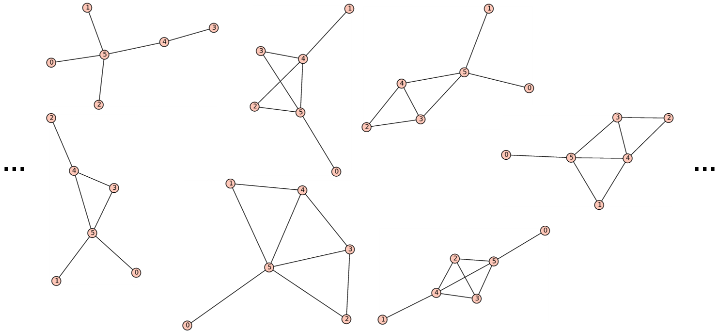
- *a dominating vertex, or*
- *an isolated vertex**, or***
- ***a weakly dominating vertex, or***
- ***a weakly isolated vertex, or***
- ***a semi-joined** P_4**.***







Non-threshold, weakly threshold graphs



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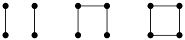
A forbidden subgraph characterization

G is a threshold graph iff G is $\{2K_2, P_4, C_4\}$-free



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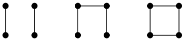
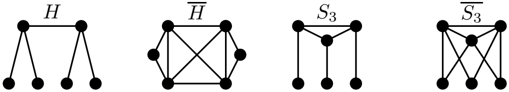


Theorem

The class of weakly threshold graphs is closed under taking induced subgraphs.
In fact, a graph G is weakly threshold if and only if it is $\{2K_2, C_4, C_5, H, \overline{H}, S_3, \overline{S_3}\}$-free.



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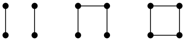

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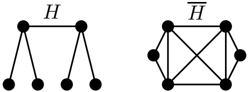
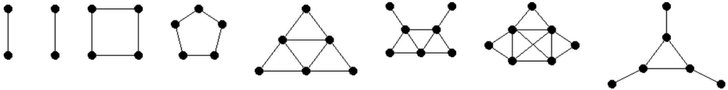


Weakly threshold graphs form a large subclass of **interval** $\cap$ **co-interval**.

(The latter class's forbidden induced subgraphs:)



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- **Iterative construction** ☑

- How many weakly threshold sequences/graphs on n vertices?
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- ...?...

Enumeration: more subtle

Threshold iff constructed from • via dominating/isolated vertices; therefore, exactly 2^{n-1} threshold graphs on n vertices.

A graph is weakly threshold iff it is constructed from a single vertex or P_4 by iteratively adding one of ...

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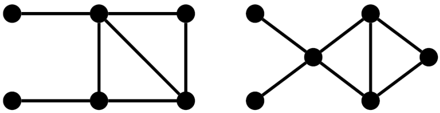
One wrinkle (of many): there is a difference between counting weakly threshold sequences / weakly threshold graphs (isomorphism classes).



Unlike threshold sequences, some weakly threshold sequences have multiple realizations!

Enumeration: sequences

a_n = number of weakly threshold **sequences** of length n

Proposition: For all $n \geq 4$, $a_n = 4a_{n-1} - 4a_{n-2} + a_{n-4}$.

$$(1,)1, 2, 4, 9, 21, 50, 120, 289, 697, 1682, 4060, \ldots$$

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It's in OEIS.org! Sequences A024537, A171842

- Binomial transform of 1, 0, 1, 0, 2, 0, 4, 0, 8, 0, 16, . . .
- Number of nonisomorphic n-element interval orders with no 3-element antichain.
- Top left entry of the nth power of $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ or of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
- Number of $(1, s_1, \ldots, s_{n-1}, 1)$ such that $s_i \in \{1, 2, 3\}$ and $|s_i - s_{i-1}| \leq 1$.
- Partial sums of the Pell numbers prefaced with a 1.
- The number of ways to write an $(n-1)$-bit binary sequence and then give runs of ones weakly incrementing labels starting with 1, e.g., 00110100110222030033300440400555555.
- Lower bound of the order of the set of equivalent resistances of $(n-1)$ equal resistors combined in series and in parallel.

Enumeration: graphs

b_n = number of weakly threshold **graphs** with n vertices

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Theorem

The generating function for (b_n) is given by

$$\sum_{n=0}^{\infty} b_n x^n = \frac{x - 2x^2 - x^3 - x^5}{1 - 4x + 3x^2 + x^3 + x^5}.$$

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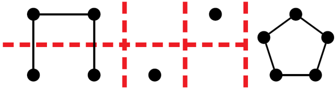

$h_n = \#$ indecomposable WT on n vertices $\qquad h_n = 3h_{n-1} - h_{n-2}$

$$H(x) = 2x + \sum_{k=4}^{\infty} h_n(x) = \frac{2x - 6x^2 + 2x^3 + x^4 - x^5 + x^6}{1 - 3x + x^2}$$

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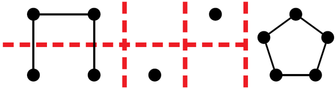

WT graphs with exactly k canonical components: $H(x)^{k-1}(H(x) - x)$

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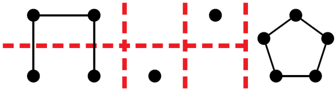

WT graphs with exactly k canonical components: $H(x)^{k-1}(H(x) - x)$

$$\sum_{n=0}^{\infty} b_n(x) = \sum_{k=1}^{\infty} H(x)^{k-1}(H(x) - x) = \frac{H(x) - x}{1 - H(x)}$$

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$$\sum_{n=0}^{\infty} b_n x^n = \frac{x - 2x^2 - x^3 - x^5}{1 - 4x + 3x^2 + x^3 + x^5}.$$

Enumeration: graphs

b_n = number of weakly threshold **graphs** with n vertices

Theorem

The generating function for (b_n) is given by

$$\sum_{n=0}^{\infty} b_n x^n = \frac{x - 2x^2 - x^3 - x^5}{1 - 4x + 3x^2 + x^3 + x^5}.$$

$$\begin{aligned} b_n = & c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \\ & + c_3 \left(\frac{6 - (1+i\sqrt{3})(27-3\sqrt{57})^{1/3} - (1-i\sqrt{3})(27+3\sqrt{57})^{1/3}}{6}\right)^n \\ & + c_4 \left(\frac{6 - (1-i\sqrt{3})(27-3\sqrt{57})^{1/3} - (1+i\sqrt{3})(27+3\sqrt{57})^{1/3}}{6}\right)^n \\ & + c_5 \left(\frac{3 + (27-3\sqrt{57})^{1/3} + (27+3\sqrt{57})^{1/3}}{3}\right)^n, \end{aligned}$$

Enumeration

There are exactly $\frac{1}{2} \cdot 2^n$ threshold graphs on n vertices.

$$a_n \sim \frac{1}{4}(1+\sqrt{2})^n$$

and

$$b_n \sim c_5 \left(\frac{3 + (27 - 3\sqrt{57})^{1/3} + (27 + 3\sqrt{57})^{1/3}}{3} \right)^n,$$

so for large n,

$$a_n \geq \frac{1}{4} \cdot 2.4^n \quad \text{and} \quad b_n \geq 0.096 \cdot 2.7^n.$$

Weakly threshold sequences and graphs

- Equality or a difference of 1 in each of the first $m(d)$ Erdős–Gallai inequalities.

  $k(k-1) + \sum_{i>k} \min\{k, d_i\} - \sum_{i \leq k} d_i \leq 1$

  Call these **weakly threshold sequences**; call the associated graphs **weakly threshold graphs**.

- **Iterative construction** ☑

- **How many weakly threshold sequences/graphs on n vertices?** ☑

- **Forbidden subgraph characterization** ☑

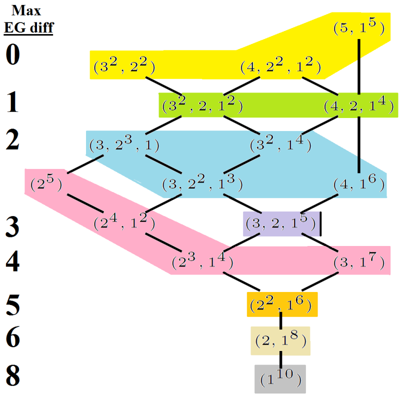
- **Unique realizations of degree sequences?** **NO**

- **Majorization result** ☑

- **...?...**

Further questions

Many of the results for weakly threshold graphs appear to generalize to graphs with $\mathrm{RHS}_k(d) - \mathrm{LHS}_k(d)$ bounded by b. Do they all?



What else can be said about graphs with near-equality in the Erdős–Gallai inequalities?

Thank you!

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