Erdős–Gallai near-equalities and the graphs that exhibit them

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Degree sequences

\[ d(G) = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2) \]
Degree sequences

\[ d(G) = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2) \]
Degree sequences

\[ d(G) = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2) \]
Degree sequences (?)

\[ d(G) = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2) \]
A key criterion

The Erdős–Gallai inequalities (1960)

A list \((d_1, \ldots, d_n)\) of nonnegative integers in descending order with even sum is a degree sequence if and only if

\[
\sum_{i \leq k} d_i \leq k(k - 1) + \sum_{i > k} \min\{k, d_i\}
\]

for all \(k\).

\((4,3,1,1,1)\) \quad (3,2,2,2,1)
A key criterion

**The Erdős–Gallai inequalities** (1960) See also Hammer–Ibaraki–Simeone (1978)

A list \((d_1, \ldots, d_n)\) of nonnegative integers in descending order with even sum is a degree sequence if and only if

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\]

for all \(k \leq \max\{i : d_i \geq i - 1\}\).

\((4,3,1,1,1)\) \hspace{2cm} (3,2,2,2,1)
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\((4,3,1,1,1)\) \hspace{1cm} (3,2,2,2,1)

\[4 < 1 \cdot 0 + 1 + 1 + 1 + 1\]

\[7 > 2 \cdot 1 + 1 + 1 + 1\]
A key criterion

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\[(4,3,1,1,1)\]

\[
4 < 1 \cdot 0 + 1 + 1 + 1 + 1
\]

\[
7 > 2 \cdot 1 + 1 + 1 + 1
\]

\[(3,2,2,2,1)\]

\[
3 < 1 \cdot 0 + 1 + 1 + 1 + 1
\]

\[
5 < 2 \cdot 1 + 2 + 2 + 1
\]

\[
7 < 3 \cdot 2 + 2 + 1
\]
Chapter 1: Spotting Erdős–Gallai *Equalities*

\[
\sum_{i \leq k} d_i = k(k - 1) + \sum_{i > k} \min\{k, d_i\}
\]
Why \[ \sum_{i \leq k} d_i \leq k(k - 1) + \sum_{i > k} \min\{k, d_i\} \]
Why \( \sum_{i \leq k} d_i \leq k(k - 1) + \sum_{i > k} \min\{k, d_i\} \)

\[ \sum_{i \leq k} d_i - k(k - 1) \text{: Lower bound on number of edges between } A \text{ and } B \]

\[ \sum_{i > k} \min\{k, d_i\} \text{: Upper bound on number of edges between } A \text{ and } B \]
What if equality holds?

Assuming $k \leq \max\{i : d_i \geq i - 1\}$

$$\sum_{i \leq k} d_i - k(k - 1) = \# \text{ edges leaving } A = \sum_{i > k} \min\{k, d_i\}$$
What if equality holds?
Assuming \( k \leq \max\{i : d_i \geq i - 1\} \)

\[
\sum_{i \leq k} d_i - k(k - 1) = \text{# edges leaving } A = \sum_{i > k} \min\{k, d_i\}
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What if equality holds?
Assuming $k \leq \max\{i : d_i \geq i - 1\}$

$$\sum_{i \leq k} d_i - k(k - 1) = \# \text{ edges leaving } A = \sum_{i > k} \min\{k, d_i\}$$

Clique

$v_1, \ldots, v_k$

A

degrees less than $k$

B

degrees at least $k$

C

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Erdős–Gallai near-equalities

April 5, 2017
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Assuming $k \leq \max\{i : d_i \geq i - 1\}$

$$\sum_{i \leq k} d_i - k(k-1) = \# \text{ edges leaving } A = \sum_{i > k} \min\{k, d_i\}$$

Clique

$v_1, \ldots, v_k$

$A$

degrees at least $k$

No edges from $B$ to $C$

$B$

degrees less than $k$

Independent set
What if equality holds?

Assuming \( k \leq \max\{i : d_i \geq i - 1\} \)

\[
\sum_{i \leq k} d_i - k(k - 1) = \text{# edges leaving } A = \sum_{i > k} \min\{k, d_i\}
\]

Clique

\( v_1, \ldots, v_k \)

\( A \)

All edges from \( A \) to \( C \)

degrees at least \( k \)

\( B \)

No edges from \( B \) to \( C \)

degrees less than \( k \)

Independent set
First sightings

**Split graphs** (Hammer–Simeone, 1981)

\[
\sum_{i=1}^{m} d_i = m(m - 1) + \sum_{i=m+1}^{n} d_i,
\]

where \( m = \max\{i : d_i \geq i - 1\} \).

**Pseudo-split graphs** (Blázsik et al., 1993; Maffray–Preissmann, 1994)

\[
\sum_{i=1}^{m} d_i = m(m - 1) + 5m + \sum_{i=m+1}^{n} d_i \quad \text{and} \quad d_{m+1} = d_{m+2} = d_{m+3} = d_{m+4} = d_{m+5} = m + 2,
\]

where \( m = \max\{i : d_i \geq i - 1\} \).
What if more than one equality holds?

Iterated partitioning

For $d = (9, 9, 7, 7, 6, 6, 6, 3, 3, 1, 1)$,

\[
9 + 9 = 2 \cdot 1 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 1 + 1
\]

\[
9 + 9 + 7 + 7 = 4 \cdot 3 + 4 + 4 + 4 + 3 + 3 + 1 + 1
\]
What if more than one equality holds?

Iterated partitioning

For \( d = (9, 9, 7, 7, 6, 6, 6, 3, 3, 1, 1) \),

\[
9 + 9 = 2 \cdot 1 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 1 + 1
\]

\[
9 + 9 + 7 + 7 = 4 \cdot 3 + 4 + 4 + 4 + 3 + 3 + 1 + 1
\]
Classes with multiple equalities

**Matrogenic/matroidal graphs** (Tyshkevich, 1984; see also B, 2013)

For each consecutive pair $k, k'$ of indices with EG-equality with $k \geq k + 2$,

- terms $d_i$ with $k < i \leq k'$ all equal, in $\{d_{k'}^*, d_{k}^* - \delta_k - 2\}$
- terms $d_i$ with $k < d_i < k'$ all equal, in $\{k + 1, k' - 1\}$

Terms past the last index $t$ of EG-equality, with $d_i > t$, collectively form one of $((t + 1)^{2r}), ((t + 2r - 2)^{2r}), ((t + 2)^{5r})$.

**Threshold graphs**
(Hammer–Ibaraki–Simeone, 1978)

$$\sum_{i=1}^{k} d_i = k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$$

for all $k \leq \max\{i : d_i \geq i - 1\}$. 
Tyshkevich’s “canonical decomposition” (∼ 1980, 2000)

Theorem

Every graph $F$ can be represented uniquely as a composition

$$F = (G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ F_0$$

of indecomposable components.

($G_i, A_i, B_i$): indecomposable splitted graphs
$F_0$: indecomposable graph

(B, 2013) Partition can be recognized via Erdős–Gallai equalities.
Other graph families with EG-equality connections 
via the canonical decomposition


**Box-threshold graphs** (Tyshkevich–Chernyak, 1985)

**Hereditary unigraphs** (B, 2013)

**Decisive graphs** (B, 2014)

Most interesting families with characterizations purely in terms of a degree sequence?
A number of interesting graph classes have degree sequence characterizations that can be rephrased as having one or more Erdős–Gallai inequalities hold with equality.
Chapter 2: Why are we studying Erdős–Gallai equalities?

\[ \sum_{i \leq k} d_i = k(k - 1) + \sum_{i > k} \min\{k, d_i\} \]
A question

Are there any edges or non-edges *forced* by the degree sequence?

A question

Are there any edges or non-edges \textit{forced} by the degree sequence?

\[ d(G) = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 0) \]
Forcible edges and/or non-edges?

\[ d = (2, 2, 2, 1, 1) \]

\( d = (4, 3, 2, 2, 1) \)

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Forcible edges and/or non-edges?

\[ d = (2, 2, 2, 1, 1) \]

(Hammer–Ibaraki–Simeone, 1978) Threshold graphs are the unique labeled realizations of their degree sequences—every edge, non-edge is forced by the degree sequence

\[ d = (4, 3, 2, 2, 1) \]

\[ d = (2, 2, 1, 1) \]
Forcible edges and non-edges: results

Notation: \[ \sum_{i \leq k} d_i \leq k(k - 1) + \sum_{i > k} \min\{k, d_i\} \]

LHS\(_k\)(\(d\)) \[ \leq \] \(k\) \((k - 1) + \sum_{i > k} \min\{k, d_i\} \]

RHS\(_k\)(\(d\))

Theorem (B, 2017+)

Given 1 \(\leq\) \(i\) < \(j\) \(\leq\) \(n\),

\(v_i v_j\) is a **forced edge** iff \(\exists k \in \{1, \ldots, n\}\) such that

*either* \(\text{LHS}_k(d) = \text{RHS}_k(d)\), \(i \leq k < j\), and \(k \leq d_j\),

*or* \(\text{LHS}_k(d) + 1 = \text{RHS}_k(d)\) and \(j \leq k\).

\(v_i v_j\) is a **forced non-edge** iff \(\exists k \in \{1, \ldots, n\}\) such that

*either* \(\text{LHS}_k(d) = \text{RHS}_k(d)\), \(k < i\), and \(d_j < k \leq d_i\),

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Forcible edges and non-edges: results

Notation: \( \sum_{i \leq k} d_i \leq k(k - 1) + \sum_{i > k} \min\{k, d_i\} \)

**Theorem (B, 2017+)**

Given \( 1 \leq i < j \leq n \),

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- or \( \text{LHS}_k(d) + 1 = \text{RHS}_k(d) \) and \( d_i < k < i \).
Looking beyond equality, some interesting things can happen when $\text{LHS}_k(d) \approx \text{RHS}_k(d)$. 
Chapter 3: Pushing the Boundaries on Threshold Graphs

\[ \sum_{i \leq k} d_i \approx k(k - 1) + \sum_{i > k} \min\{k, d_i\} \]
Threshold graphs

Two definitions

Hammer–Ibaraki–Simeone, 1978:

\[ \sum_{i=1}^{k} d_i = k(k - 1) + \sum_{i=k+1}^{n} \min\{k, d_i\} \]

for all \( k \leq \max\{i : d_i \geq i - 1\} \).

Chvátal–Hammer, 1973:

Weights on vertices, threshold for adjacency

Threshold: 4.5

\[ \begin{array}{cccccc}
\text{1} & \text{2} & \text{3} & \text{4} \\
\text{1} & \text{2} & \text{3} & \text{4} \\
\end{array} \]
Other properties of threshold graphs
(Chvátal, Hammer, others, 1973+)

- Equality in the first \( m(d) \) Erdős–Gallai inequalities.
  \[
  \sum_{i \leq k} d_i = k(k - 1) + \sum_{i > k} \min\{k, d_i\}
  \]

- Iterative construction via dominating/isolated vertices

- There are exactly \( 2^{n-1} \) threshold graphs on \( n \) vertices.

- \( \{2K_2, P_4, C_4\} \)-free

- Unique realization of degree sequence

- Threshold sequences majorize all other degree sequences
Threshold graph building

$G$ is a threshold graph if and only if $G$ can be constructed from a single vertex via these operations. Consequently, up to isomorphism there are exactly $2^n - 1$ threshold graphs on $n$ vertices.
Threshold graph building

Options for adding

- Dominating vertex
- Isolated vertex

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Consequently, up to isomorphism there are exactly $2^{n-1}$ threshold graphs on $n$ vertices.
Equality everywhere: properties of threshold graphs
(Chvátal, Hammer, others, 1973+)

- Equality in the first \( m(d) \) Erdős–Gallai inequalities.
  \[ \sum_{i \leq k} d_i = k(k - 1) + \sum_{i > k} \min\{k, d_i\} \]

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- There are exactly \( 2^{n-1} \) threshold graphs on \( n \) vertices.

- \( \{2K_2, P_4, C_4\} \)-free

- Unique realization of degree sequence

- Threshold sequences majorize all other degree sequences...
A forbidden subgraph characterization

**Induced subgraph:** a subgraph obtained by deleting vertices and their incident edges

![Graphs](image)

**Theorem (Chvátal–Hammer, 1973)**

Any induced subgraph of a threshold graph is a threshold graph. In fact, $G$ is a threshold graph iff $G$ has no induced subgraph isomorphic to one of the following:

![Graphs](image)

(We say that $G$ is $\{2K_2, P_4, C_4\}$-free.)
Threshold sequences and majorization

Majorization:
Given \(d = (d_1, \ldots, d_n)\) and \(e = (e_1, \ldots, e_m)\),
\(d \succeq e\) if \(\sum d_i = \sum e_i\)
and
\[
\sum_{i=1}^k d_i \geq \sum_{i=1}^k e_i \text{ for all } k.
\]

Theorem (Ruch–Gutman, 1979; Peled–Srinivasan, 1989)
\(d\) is a threshold sequence if and only if \(d\) is a maximal element in the poset of all degree sequences with the same sum, ordered by majorization.
Properties of threshold graphs
(Chvátal, Hammer, others, 1973+)

- Equality in the first $m(d)$ Erdős–Gallai inequalities.
  \[ \sum_{i \leq k} d_i = k(k - 1) + \sum_{i > k} \min\{k, d_i\} \]
- Iterative construction via dominating/isolated vertices
- There are exactly $2^{n-1}$ threshold graphs on $n$ vertices.

\{2K_2, P_4, C_4\}-free

Unique realization of degree sequence

Threshold sequences majorize all other degree sequences

...
What if we required Erdős–Gallai near-equality?

Define a **weakly threshold sequence** to be list \( d = (d_1, \ldots, d_n) \) of nonnegative integers in descending order having even sum and satisfying

\[
RHS_k(d) - 1 \leq LHS_k(d) \leq RHS_k(d)
\]

for all \( k \leq \max\{i : d_i \geq i - 1\} \).
What if we required Erdős–Gallai near-equality?

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for all \( k \leq \max\{i : d_i \geq i - 1\} \).

A **weakly threshold graph** will be a graph having a weakly threshold sequence as its degree sequence.
Weakly threshold sequences and graphs
(B, 2017+)

- Equality or a difference of 1 in each of the first $m(d)$ Erdős–Gallai inequalities.
  \[ k(k - 1) + \sum_{i > k} \min\{k, d_i\} - \sum_{i \leq k} d_i \leq 1 \]

Call these weakly threshold sequences; call the associated graphs weakly threshold graphs.

- How many weakly threshold sequences/graphs on $n$ vertices?
- Forbidden subgraph characterization?
- Unique realizations of degree sequences?
- Majorization result?
  ...?...
Near the threshold \((B, 2017+)\)

Threshold sequences majorize all other degree sequences.
Near the threshold \((B, 2017+)\)

Threshold sequences majorize all other degree sequences.

WT sequences (and all diff \(\leq b\)) are upwards-closed, continue to majorize.
Weakly threshold sequences and graphs

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Call these **weakly threshold sequences**; call the associated graphs **weakly threshold graphs**.

- Iterative construction?

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- Forbidden subgraph characterization?

- Unique realizations of degree sequences?

- **Majorization result**

...?...
Theorem

A graph $G$ is a threshold graph if and only if $G$ can be constructed by beginning with a single vertex and iteratively adding:

- a dominating vertex, or
- an isolated vertex.
Theorem

$G$ is a **weakly** threshold graph if and only if $G$ can be constructed by beginning with a single vertex **or** $P_4$ and iteratively adding

- a dominating vertex, **or**
- an isolated vertex, **or**
- a weakly dominating vertex, **or**
- a weakly isolated vertex, **or**
- a semi-joined $P_4$. 

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Erdős–Gallai near-equalities  
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Non-threshold, weakly threshold graphs
Weakly threshold sequences and graphs

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\[ k(k - 1) + \sum_{i > k} \min\{k, d_i\} - \sum_{i \leq k} d_i \leq 1 \]

Call these **weakly threshold sequences**; call the associated graphs **weakly threshold graphs**.

Iterative construction

- How many weakly threshold sequences/graphs on $n$ vertices?
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...?...
A forbidden subgraph characterization

$G$ is a threshold graph iff $G$ is $\{2K_2, P_4, C_4\}$-free

Theorem

The class of weakly threshold graphs is closed under taking induced subgraphs. In fact, a graph $G$ is weakly threshold if and only if it is $\{2K_2, C_4, C_5, H, H, S_3, S_3\}$-free.
A forbidden subgraph characterization

G is a threshold graph iff G is \{2K_2, P_4, C_4\}-free

**Theorem**

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In fact, a graph G is weakly threshold if and only if it is \{2K_2, C_4, C_5, H, \overline{H}, S_3, \overline{S_3}\}-free.
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**Theorem**

The class of weakly threshold graphs is closed under taking induced subgraphs.

In fact, a graph $G$ is weakly threshold if and only if it is $\{2K_2, C_4, C_5, H, \overline{H}, S_3, \overline{S_3}\}$-free.

Weakly threshold graphs form a large subclass of interval $\cap$ co-interval.

(The latter class's forbidden induced subgraphs:)

M. D. Barrus (URI) Erdős–Gallai near-equalities April 5, 2017
Weakly threshold sequences and graphs

- Equality or a difference of 1 in each of the first $m(d)$ Erdős–Gallai inequalities.
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  Call these **weakly threshold sequences**; call the associated graphs **weakly threshold graphs**.

- Iterative construction

- How many weakly threshold sequences/graphs on $n$ vertices?

- **Forbidden subgraph characterization**

- Unique realizations of degree sequences?

- **Majorization result**

...?...
Enumeration: more subtle

Threshold iff constructed from \( \bullet \) via dominating/isolated vertices; therefore, exactly \( 2^{n-1} \) threshold graphs on \( n \) vertices.

A graph is weakly threshold iff it is constructed from a single vertex or \( P_4 \) by iteratively adding one of ...
Enumeration: more subtle

Threshold iff constructed from $\bullet$ via dominating/isolated vertices; therefore, exactly $2^{n-1}$ threshold graphs on $n$ vertices.

A graph is weakly threshold iff it is constructed from a single vertex or $P_4$ by iteratively adding one of ...

One wrinkle (of many): there is a difference between counting weakly threshold sequences / weakly threshold graphs (isomorphism classes).

Unlike threshold sequences, some weakly threshold sequences have multiple realizations!
Enumeration: sequences

\( a_n = \text{number of weakly threshold sequences of length } n \)

**Proposition:** For all \( n \geq 4 \), \( a_n = 4a_{n-1} - 4a_{n-2} + a_{n-4} \).

\( (1, )1, 2, 4, 9, 21, 50, 120, 289, 697, 1682, 4060, \ldots \)
Enumeration: sequences

\[ a_n = \text{number of weakly threshold sequences of length } n \]

**Proposition:** For all \( n \geq 4 \),

\[ a_n = 4a_{n-1} - 4a_{n-2} + a_{n-4}. \]

\((1, )1, 2, 4, 9, 21, 50, 120, 289, 697, 1682, 4060, \ldots\)

It’s in OEIS.org! Sequences A024537, A171842

- Binomial transform of 1, 0, 1, 0, 2, 0, 4, 0, 8, 0, 16, \ldots
- Number of nonisomorphic \( n \)-element interval orders with no 3-element antichain.
- Top left entry of the \( n \)th power of \[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 1 \\
\end{pmatrix}
\] or of \[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\].
- Number of \((1, s_1, \ldots, s_{n-1}, 1)\) such that \( s_i \in \{1, 2, 3\} \) and \(|s_i - s_{i-1}| \leq 1\).
- Partial sums of the Pell numbers prefaced with a 1.
- The number of ways to write an \((n - 1)\)-bit binary sequence and then give runs of ones weakly incrementing labels starting with 1, e.g., 001101001102220330044040055555.
- Lower bound of the order of the set of equivalent resistances of \((n - 1)\) equal resistors combined in series and in parallel.
Enumeration: graphs

\[ b_n = \text{number of weakly threshold graphs with } n \text{ vertices} \]
Enumeration: graphs

\[ b_n = \text{number of weakly threshold graphs with } n \text{ vertices} \]

**Theorem**

The generating function for \((b_n)\) is given by

\[
\sum_{n=0}^{\infty} b_n x^n = \frac{x - 2x^2 - x^3 - x^5}{1 - 4x + 3x^2 + x^3 + x^5}.
\]
Enumeration: graphs

\[ b_n = \text{number of weakly threshold graphs} \] with \( n \) vertices

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The generating function for \((b_n)\) is given by

\[
\sum_{n=0}^{\infty} b_n x^n = \frac{x - 2x^2 - x^3 - x^5}{1 - 4x + 3x^2 + x^3 + x^5}.
\]

\[ h_n = \# \text{ indecomposable WT on } n \text{ vertices} \quad h_n = 3h_{n-1} - h_{n-2} \]

\[ H(x) = 2x + \sum_{k=4}^{\infty} h_n(x) = \frac{2x - 6x^2 + 2x^3 + x^4 - x^5 + x^6}{1 - 3x + x^2} \]
Enumeration: graphs

\[ b_n = \text{number of weakly threshold graphs with } n \text{ vertices} \]

**Theorem**

The generating function for \((b_n)\) is given by

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\sum_{n=0}^{\infty} b_n x^n = \frac{x - 2x^2 - x^3 - x^5}{1 - 4x + 3x^2 + x^3 + x^5}.
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# WT graphs with exactly \(k\) canonical components: \(H(x)^{k-1}(H(x) - x)\)
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# WT graphs with exactly \(k\) canonical components: \(H(x)^{k-1}(H(x) - x)\)

\[
\sum_{n=0}^{\infty} b_n(x) = \sum_{k=1}^{\infty} H(x)^{k-1}(H(x) - x) = \frac{H(x) - x}{1 - H(x)}
\]
Enumeration: graphs

\[ b_n = \text{number of weakly threshold graphs with } n \text{ vertices} \]

**Theorem**

*The generating function for \((b_n)\) is given by*

\[
\sum_{n=0}^{\infty} b_n x^n = \frac{x - 2x^2 - x^3 - x^5}{1 - 4x + 3x^2 + x^3 + x^5}.
\]
Enumeration: graphs

\( b_n = \text{number of weakly threshold graphs with } n \text{ vertices} \)

**Theorem**

The generating function for \((b_n)\) is given by

\[
\sum_{n=0}^{\infty} b_n x^n = \frac{x - 2x^2 - x^3 - x^5}{1 - 4x + 3x^2 + x^3 + x^5}.
\]

\[
b_n = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n + c_3 \left( \frac{6 - (1 + i\sqrt{3})(27 - 3\sqrt{57})^{1/3} - (1 - i\sqrt{3})(27 + 3\sqrt{57})^{1/3}}{6} \right)^n + c_4 \left( \frac{6 - (1 - i\sqrt{3})(27 - 3\sqrt{57})^{1/3} - (1 + i\sqrt{3})(27 + 3\sqrt{57})^{1/3}}{6} \right)^n + c_5 \left( \frac{3 + (27 - 3\sqrt{57})^{1/3} + (27 + 3\sqrt{57})^{1/3}}{3} \right)^n,
\]
Enumeration

There are exactly $\frac{1}{2} \cdot 2^n$ threshold graphs on $n$ vertices.

$$a_n \sim \frac{1}{4}(1 + \sqrt{2})^n$$

and

$$b_n \sim c_5 \left(\frac{3 + (27 - 3\sqrt{57})^{1/3} + (27 + 3\sqrt{57})^{1/3}}{3}\right)^n,$$

so for large $n$,

$$a_n \geq \frac{1}{4} \cdot 2.4^n \quad \text{and} \quad b_n \geq 0.096 \cdot 2.7^n.$$
Weakly threshold sequences and graphs

- Equality or a difference of 1 in each of the first $m(d)$ Erdős–Gallai inequalities.
  
  $$k(k - 1) + \sum_{i > k} \min\{k, d_i\} - \sum_{i \leq k} d_i \leq 1$$

- Call these weakly threshold sequences; call the associated graphs weakly threshold graphs.

- Iterative construction

- How many weakly threshold sequences/graphs on $n$ vertices?

- Forbidden subgraph characterization

- Unique realizations of degree sequences? NO

- Majorization result

- ...?...
Further questions

Many of the results for weakly threshold graphs appear to generalize to graphs with $\text{RHS}_k(d) - \text{LHS}_k(d)$ bounded by $b$. Do they all?

What else can be said about graphs with near-equality in the Erdős–Gallai inequalities?
Thank you!

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