Three conjectures on minimal obstructions for tree-depth

Michael D. Barrus

Department of Mathematics
University of Rhode Island

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Joint work with John Sinkovic (Georgia State University)
Tree-depth \( td(G) \): The minimum number of vertex deletion steps needed to delete all of \( G \), where in each step at most one vertex is deleted from each connected component.
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(aka (vertex) ranking number, ordered coloring number, ...)

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Equivalently, the smallest number of labels needed in a labeling where every path with equal endpoints also has a higher label.
Theorem

If $G$ contains $H$ as a minor, then $td(G) \geq td(H)$. 

Tree-depth and minors
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Conjectures on tree-depth

March 3, 2015
Theorem

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Call a graph **critical** if every proper minor has a smaller tree-depth.  
($k$-critical = critical, with tree-depth $k$)

**Question:** Which are the critical graphs?
Critical graphs for small tree-depths
(Dvořák–Giannopoulou–Thilikos, ’09, ’12)

1: ● 2: ●●

What structural properties must critical graphs possess?
How are they made? Can we construct them?
Critical graphs for small tree-depths

(Dvořák–Giannopoulou–Thilikos, ’09, ’12)

1: 

2: 

3: 

What structural properties must critical graphs possess? How are they made? Can we construct them?
Critical graphs for small tree-depths
(Dvořák–Giannopoulou–Thilikos, ’09, ’12)

1: \[ \bullet \]
2: \[ \bullet - \bullet \]
3: \[ \bullet - \bullet - \bullet \]
4: \[ \text{Various graphs} \]

What structural properties must critical graphs possess?
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Critical graphs for small tree-depths
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1: \[ \bullet \]
2: \[ \bullet - \bullet \]
3: \[ \bullet - \bullet - \bullet \]
4: \[ \text{Various structures} \]
5: 136 trees, plus...
Critical graphs for small tree-depths
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1: ●
2: ●●
3: ●●●
4: ...
5: 136 trees, plus...

- What structural properties must critical graphs possess?
- How are they made? Can we construct them?
Conjectures on $k$-critical graphs

1: 

2: 

3: 

4: 

Conjecture 1 [Dvořák–Giannopoulou–Thilikos, '09, '12]

If $G$ is $k$-critical, then $|V(G)| \leq 2k - 1$. 

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Conjectures on $k$-critical graphs

1:

2:

3:

4:

Conjecture 1 [Dvořák–Giannopoulou–Thilikos, ’09, ’12]

If $G$ is $k$-critical, then

$$|V(G)| \leq 2^{k-1}.$$
Conjecture 2

If $G$ is $k$-critical, then

$$\Delta(G) \leq k - 1.$$
An approach to Conjecture 2

Any vertex with the smallest label has neighbors with distinct labels.

If in some optimal labeling of $G$ a vertex of maximum degree receives the label 1, then $\Delta(G) \leq \text{td}(G) - 1$. 
Further conjectures

1: 

2: 

3: 

4: 

Conjecture 3:
If $G$ is $k$-critical, then $G$ has an optimal labeling where some vertex with maximum degree is labeled 1.
Further conjectures

1: 

2: 

3: 

4: 

Conjecture 3(?)

If $G$ is $k$-critical, then ... $G$ has an optimal labeling where some vertex with maximum degree is labeled 1.
Further conjectures

1: 

2: 

3: 

4: 

Conjecture 3 (?)

If $G$ is $k$-critical, then for each vertex $v$ of $G$, there is an optimal labeling where $v$ receives label 1.
Further conjectures

1: 

2: 

3: 

4: 

Conjecture 3(?)
If $G$ is $k$-critical, then ... for each vertex $v$ of $G$, there is an optimal labeling where $v$ is the unique vertex receiving label 1.
1-uniqueness

Given a graph $G$, we say that a graph $G$ is 1-unique if for every vertex $v$ of $G$ there is an optimal labeling of $G$ where $v$ is the only vertex receiving label 1.

Conjecture 3 (final version)
If $G$ is critical, then $G$ is 1-unique.
Theorem

If $G$ is a 1-unique graph, then

- deleting any vertex from $G$ lowers the tree-depth;
- contracting any edge of $G$ lowers the tree-depth.
1-uniqueness—a type of criticality?

Theorem

If $G$ is a 1-unique graph, then

- deleting any vertex from $G$ lowers the tree-depth;
- contracting any edge of $G$ lowers the tree-depth.

Thus, 1-unique graphs are similar to critical graphs—each is at most some edge-deletions away from a critical graph.
1-uniqueness—a type of criticality?

Theorem

If $G$ is a 1-unique graph, then

- deleting any vertex from $G$ lowers the tree-depth;
- contracting any edge of $G$ lowers the tree-depth.

Thus, 1-unique graphs are similar to critical graphs—each is at most some edge-deletions away from a critical graph.

Is every critical graph 1-unique?
Critical graphs for small tree-depths
(Dvořák–Giannopoulou–Thilikos, ’09, ’12)

What structural properties must critical graphs possess?
How are they made? Can we construct them?
Critical graphs for small tree-depths
(Dvořák–Giannopoulou–Thilikos, ’09, ’12)

1: \[ \bullet \]
2: \[ \bullet - \bullet \]
3: \[ \bullet - \bullet - \bullet \]

4: [Diagrams of various graphs]
5: 136 trees, plus...

Theorem (DGT)
For any \( k \), adding any edge joining two critical graphs with tree-depth \( k \)
results in a critical graph with tree-depth \( k + 1 \).
A generalization

Hang -critical “appendages” off every vertex of an $\ell$-critical graph...

Are these graphs critical? (And if so, do they satisfy our conjectures?)
A generalization

Hang $k$-critical “appendages” off every vertex of an $\ell$-critical graph...
A generalization

Hang $k$-critical “appendages” off every vertex of an $\ell$-critical graph...

Are these graphs critical? (And if so, do they satisfy our conjectures?)
The key condition

Theorem

Graphs $G$ constructed with $k$-critical “appendages” $L_i$ and an $\ell$-critical “core” $H$...

- have tree-depth $k + \ell - 1$ if $H$ and all the $L_i$ are critical;
- are critical if $H$ is also 1-unique;
The key condition

Graphs $G$ constructed with $k$-critical “appendages” $L_i$ and an $\ell$-critical “core” $H$...

...have order at most $2^{\text{td}(G)-1}$ if $|V(H)| \leq 2^{\ell-1}$ and $|V(L_i)| \leq 2^{k-1}$ for all $i$.

...are 1-unique if $H$ and all the $L_i$ are 1-unique;
Families of 1-unique critical graphs

1: ●

2: ● ●

3: ● ● ●

4: ● ● ● ● ●

5: ● ● ● ●

136 trees, plus...

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Things to think about

**Fact:** Every 1-unique graph differs from a critical graph by at most the deletion of some edges.
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**Conjecture:** Every critical graph is 1-unique (and hence the “appendages” construction always produces critical graphs).
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**Conjecture:** Every critical graph is 1-unique (and hence the “appendages” construction always produces critical graphs).

**Ill-defined hope:** The critical graphs each either

- belong to one or more of a few easily defined families, or
- can be produced from smaller critical graphs via one of a few easily defined constructions.
Things to think about

1: 

2: 

3: 

4: 

Conjecture 3

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Conjecture 3

If $G$ is $k$-critical, then $G$ is 1-unique.
Things to think about

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Conjecture 1 [Dvořák–Giannopoulou–Thilikos, ’09, ’12]

If $G$ is $k$-critical, then $|V(G)| \leq 2^{k-1}$. 
Things to think about

1: 

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Conjecture 2
If $G$ is $k$-critical, then $\Delta(G) \leq k - 1$. 
Things to think about

1: ●
2: ● ●
3: ● ● ● ●
4: ● ● ● ● ●

Thank you!
Tree-depth
(aka (vertex) ranking number, ordered coloring number, ...)

Equivalently, the smallest height of a tree for which the edges of $G$ all join ancestor-descendant pairs.