Three conjectures on minimal obstructions for tree-depth

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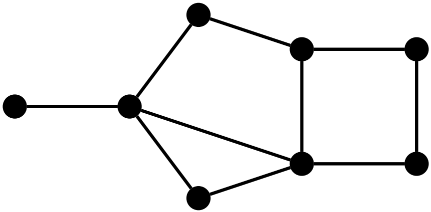
Joint work with John Sinkovic (Georgia State University)

Tree-depth

(aka (vertex) ranking number, ordered coloring number, ...)



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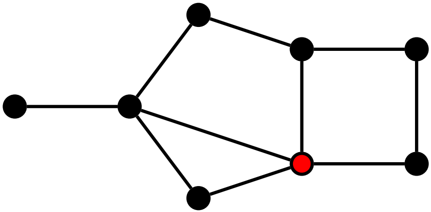

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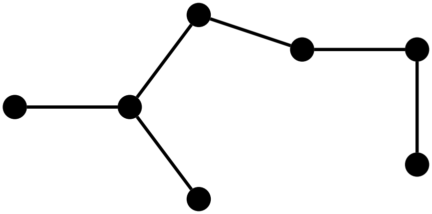

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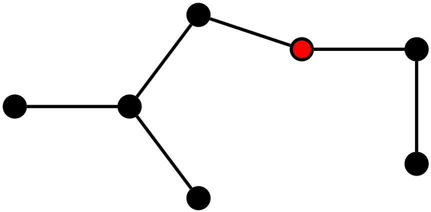

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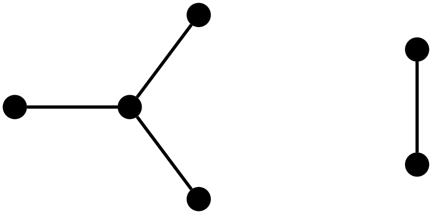

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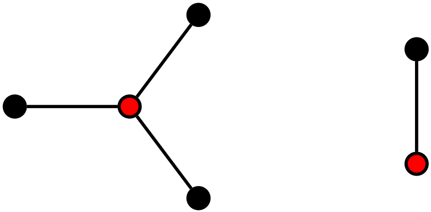

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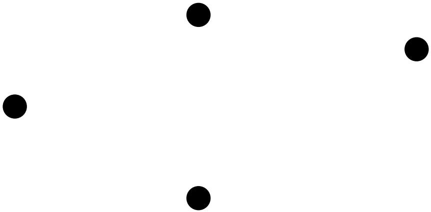

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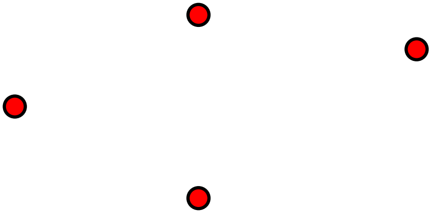

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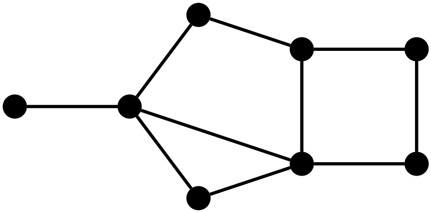

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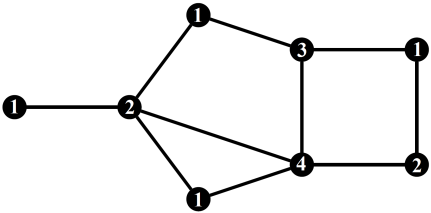
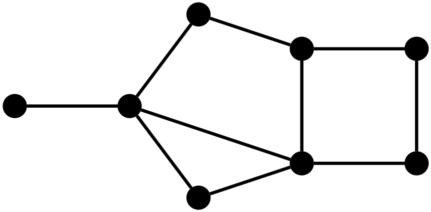


Tree-depth td(G)**:** The minimum number of vertex deletion steps needed to delete all of G, where in each step at most one vertex is deleted from each connected component. (Here, td(G) $= 4$)

Equivalently, the smallest number of labels needed in a labeling where every path with equal endpoints also has a higher label.

Tree-depth and minors



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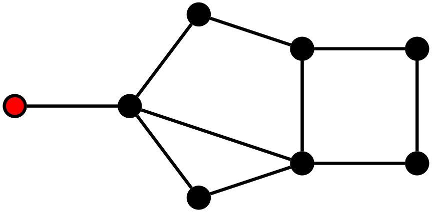

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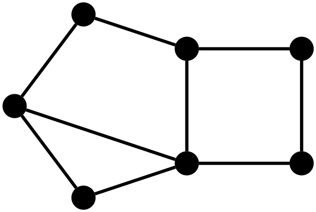

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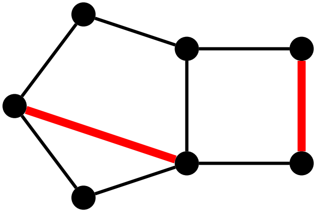

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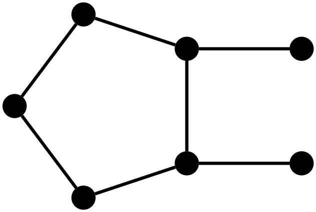

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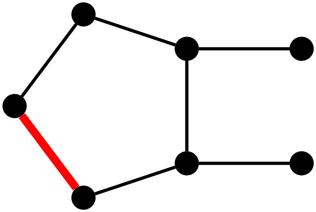

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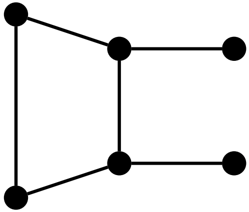

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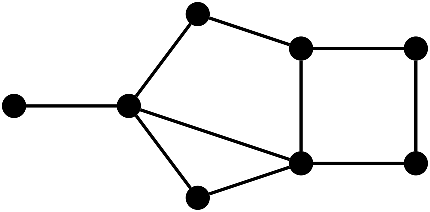
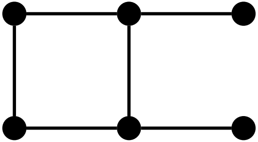
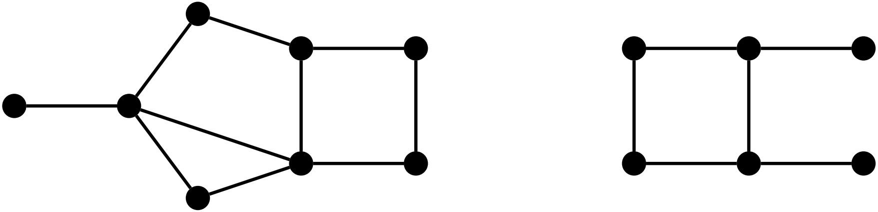




Theorem

If G contains H as a minor, then $\mathrm{td}(G) \geq \mathrm{td}(H)$.

Tree-depth and minors



Theorem

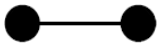
If G contains H as a minor, then $\mathrm{td}(G) \geq \mathrm{td}(H)$.

Call a graph **critical** if every proper minor has a smaller tree-depth.
(***k*-critical** = critical, with tree-depth k)

Question: Which are the critical graphs?

Critical graphs for small tree-depths

(Dvořák–Giannopoulou–Thilikos, ’09, ’12)

1:  **2:** 

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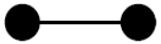
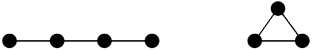
2: 

3: 

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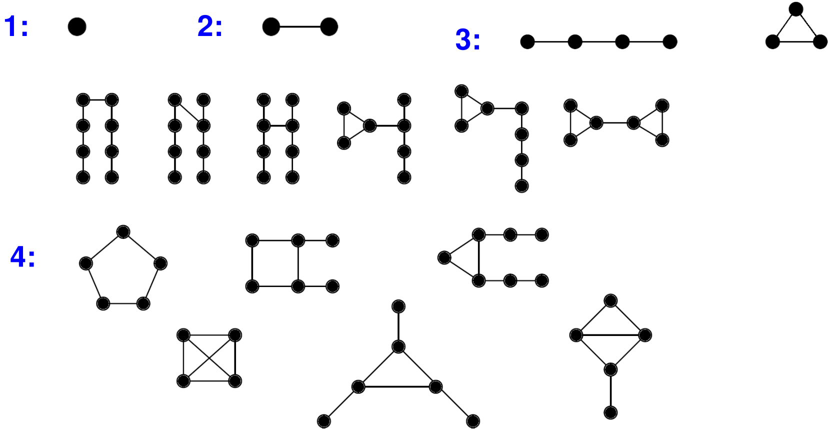

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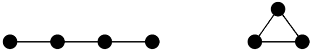
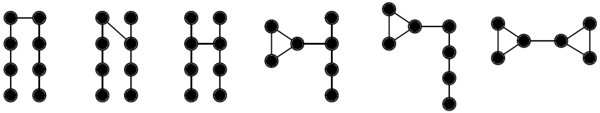
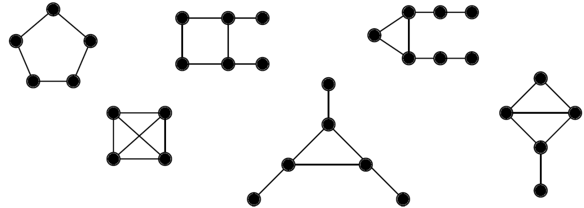
3: 



4: 

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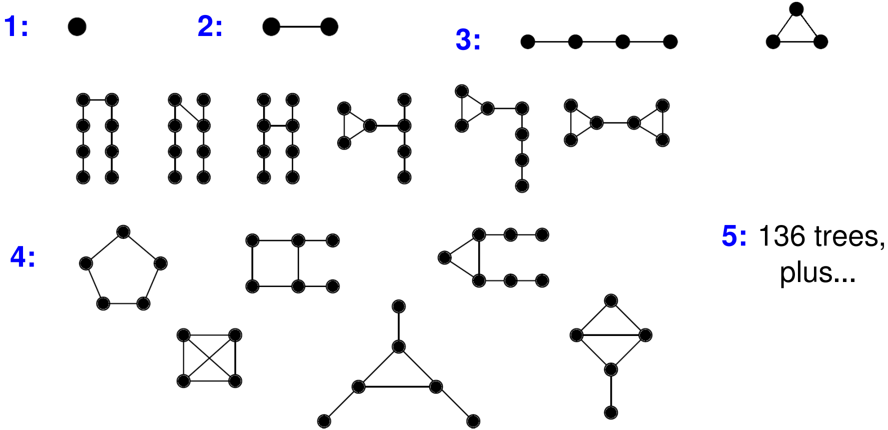
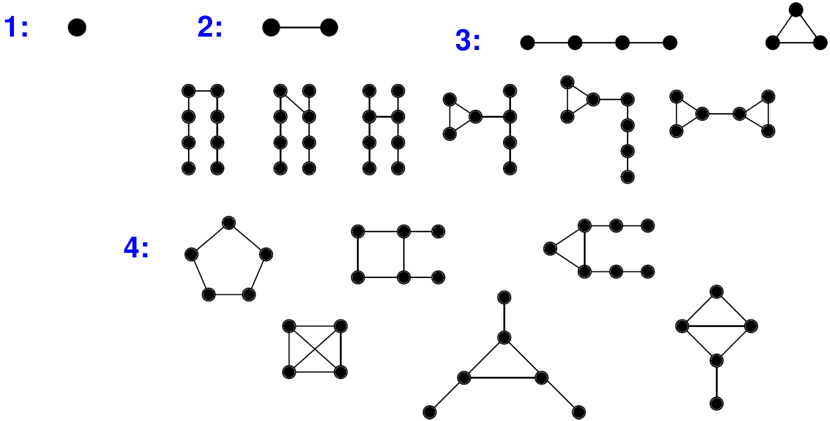
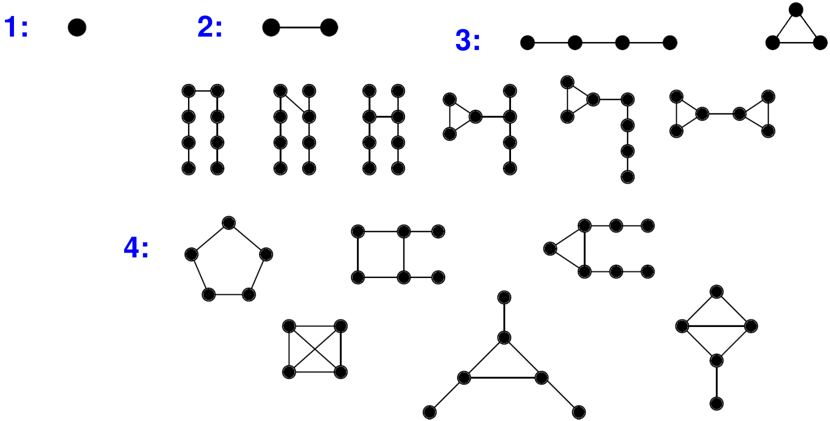


- What structural properties must critical graphs possess?
- How are they made? Can we construct them?

Conjectures on *k*-critical graphs



Conjectures on k-critical graphs

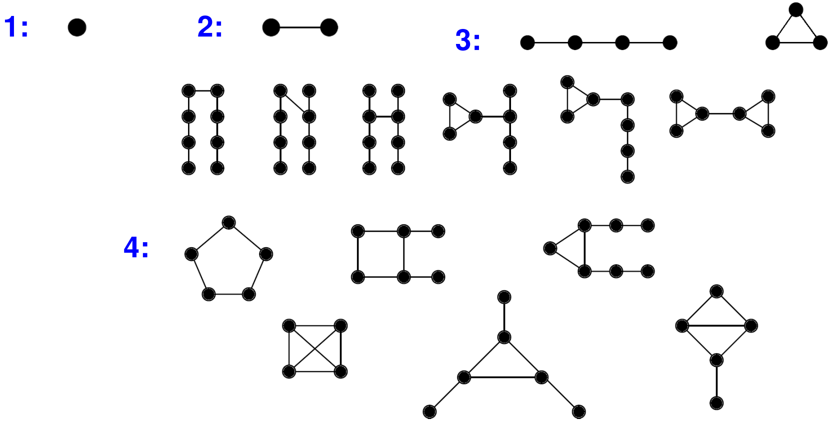


Conjecture 1 [Dvořák–Giannopoulou–Thilikos, ’09, ’12]

If G is k-critical, then

$$|V(G)| \leq 2^{k-1}.$$

Conjectures on k-critical graphs



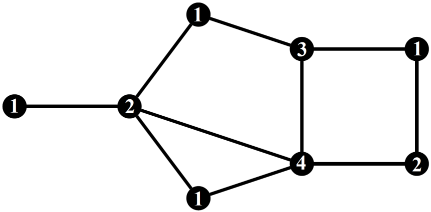
Conjecture 2

If G is k-critical, then

$$\Delta(G) \leq k - 1.$$

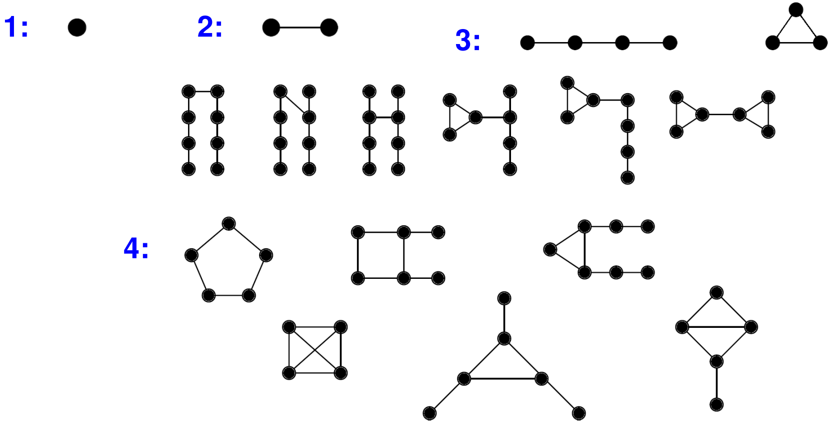
An approach to Conjecture 2

Any vertex with the smallest label has neighbors with distinct labels.



If in some optimal labeling of G a vertex of maximum degree receives the label 1, then $\Delta(G) \leq \text{td}(G) - 1$.

Further conjectures



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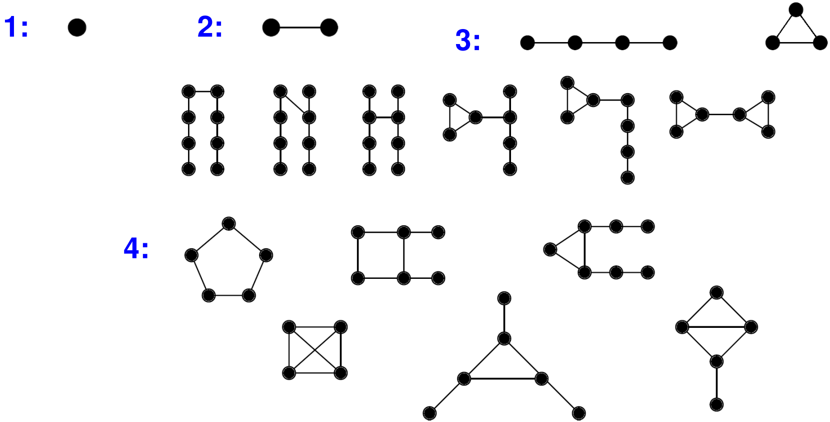
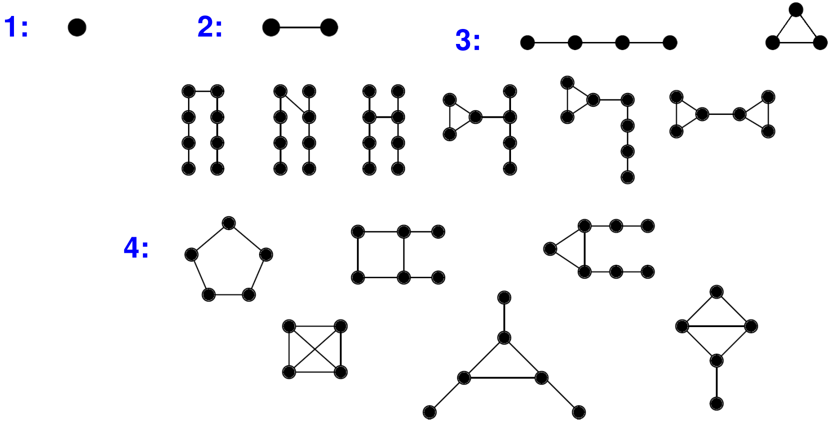


Conjecture 3(?)

If G is k-critical, then ... G has an optimal labeling where some vertex with maximum degree is labeled 1.

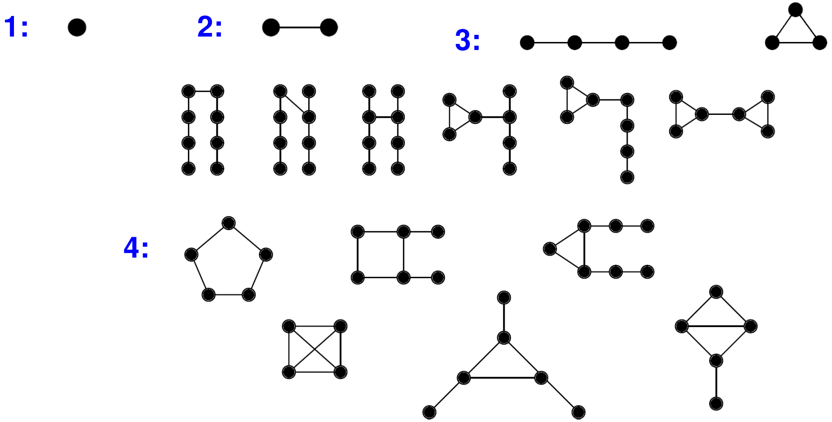
Further conjectures



Conjecture 3(?)

If G is k-critical, then ... for **each** vertex v of G, there is an optimal labeling where v receives label 1.

Further conjectures

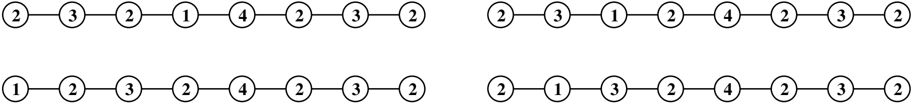


Conjecture 3(?)

If G is k-critical, then ... for **each** vertex v of G, there is an optimal labeling where v is the **unique** vertex receiving label 1.

1-uniqueness

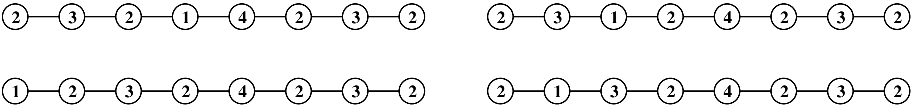
Given a graph G, we say that a graph G is **1-unique** if for every vertex v of G there is an optimal labeling of G where v is the only vertex receiving label 1.



Conjecture 3 (final version)

If G is critical, then G is 1-unique.

1-uniqueness—a type of criticality?



Theorem

If G is a 1-unique graph, then

- *deleting any vertex from G lowers the tree-depth;*
- *contracting any edge of G lowers the tree-depth.*

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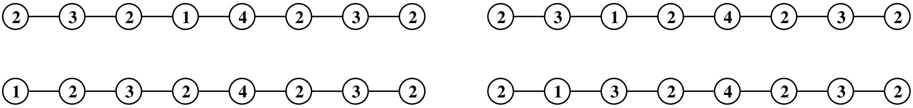

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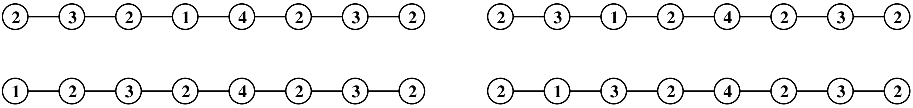

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Is every critical graph 1-unique?

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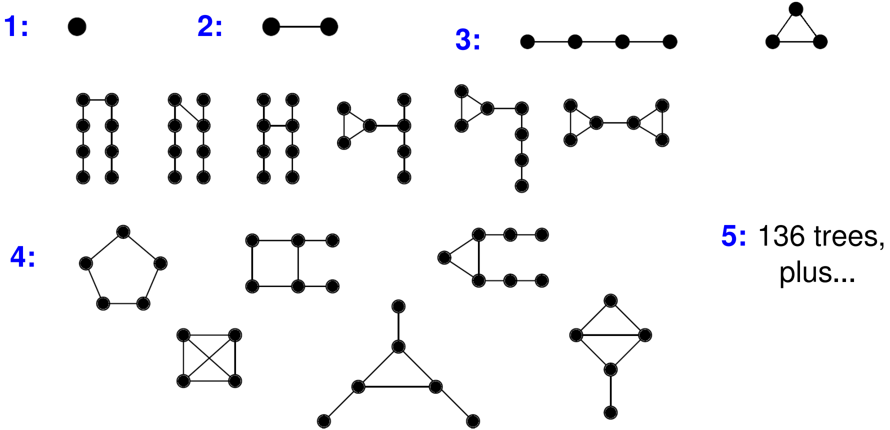

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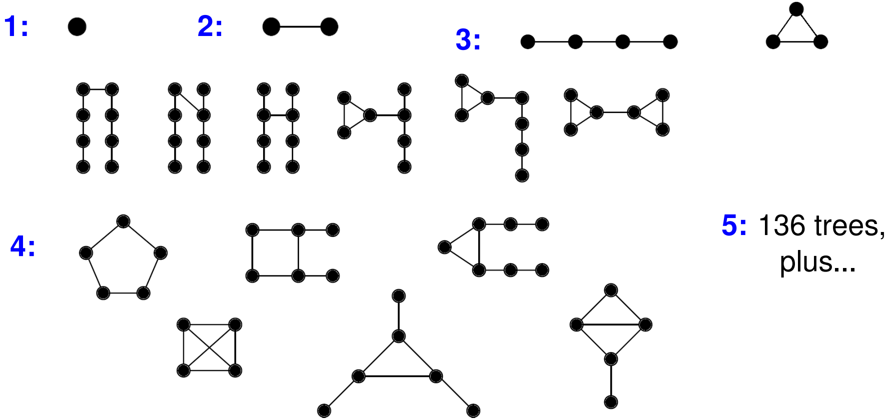
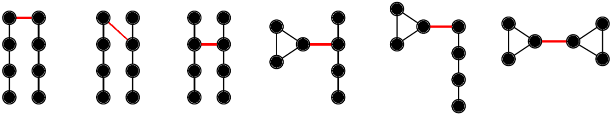


Theorem (DGT)

For any k, adding any edge joining two critical graphs with tree-depth k results in a critical graph with tree-depth $k+1$.

A generalization



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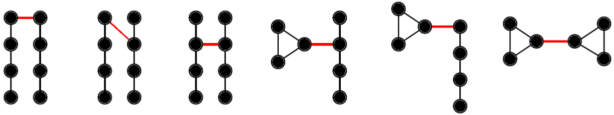
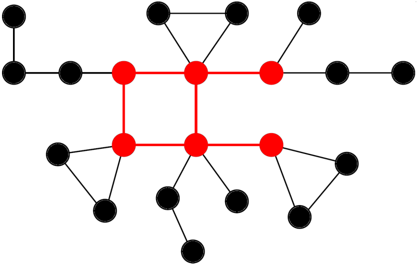


Hang k-critical “appendages” off every vertex of an ℓ-critical graph...



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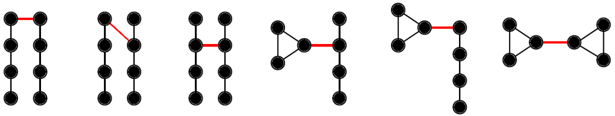

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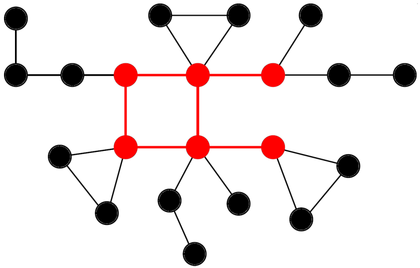
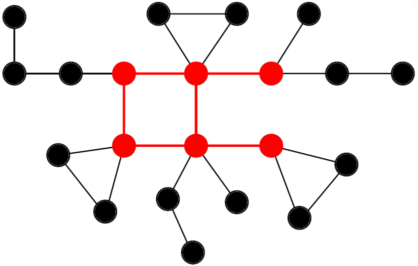


Are these graphs critical? (And if so, do they satisfy our conjectures?)

The key condition

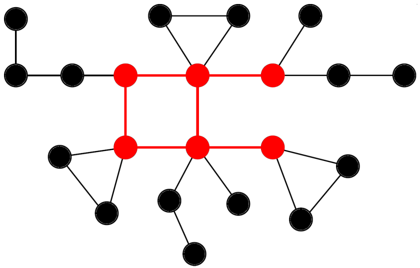


Theorem

Graphs G constructed with k-critical "appendages" L_i and an ℓ-critical "core" H...

- *...have tree-depth $k + \ell - 1$ if H and all the L_i are critical;*
- *...are critical if H is also 1-unique;*

The key condition



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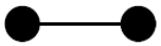
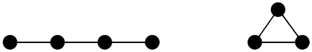
Graphs G constructed with k-critical "appendages" L_i and an ℓ-critical "core" H...

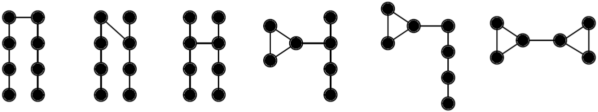
- *...have order at most $2^{\mathrm{td}(G)-1}$ if $|V(H)| \leq 2^{\ell-1}$ and $|V(L_i)| \leq 2^{k-1}$ for all i.*
- *...are 1-unique if H and all the L_i are 1-unique;*

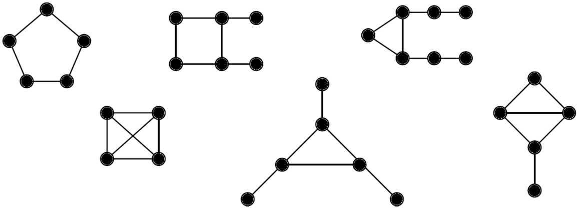
Families of 1-unique critical graphs

1: 

2: 

3: 



4: 

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Things to think about

Fact: Every 1-unique graph differs from a critical graph by at most the deletion of some edges.

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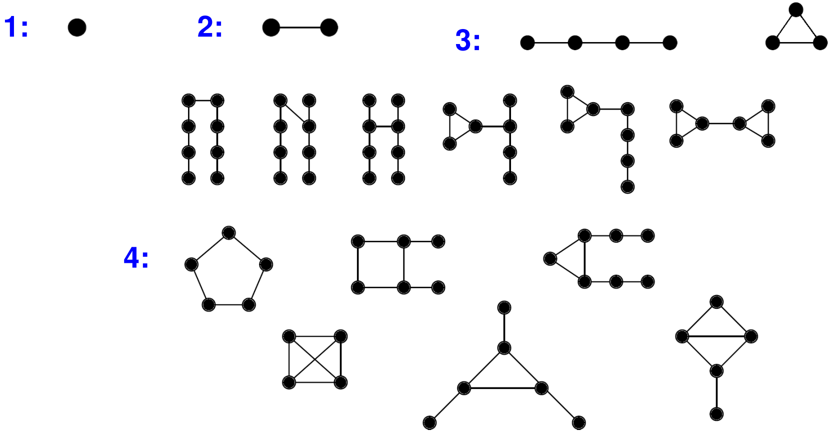
Conjecture: Every critical graph is 1-unique (and hence the "appendages" construction always produces critical graphs).

Ill-defined hope: The critical graphs each either

- belong to one or more of a few easily defined families, or
- can be produced from smaller critical graphs via one of a few easily defined constructions.

Things to think about



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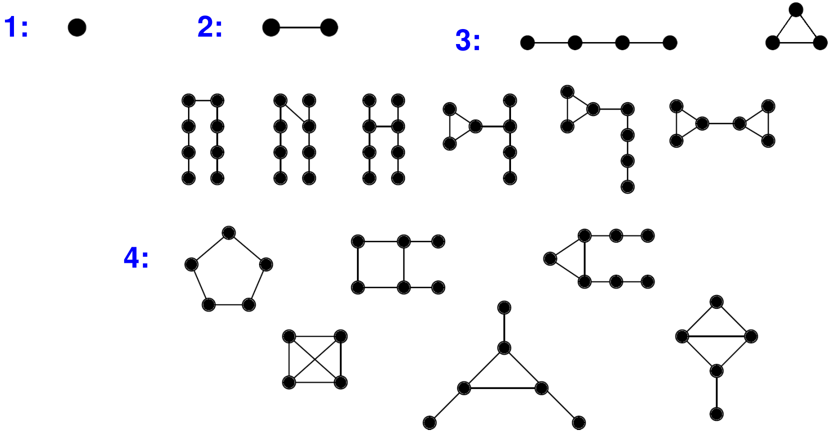
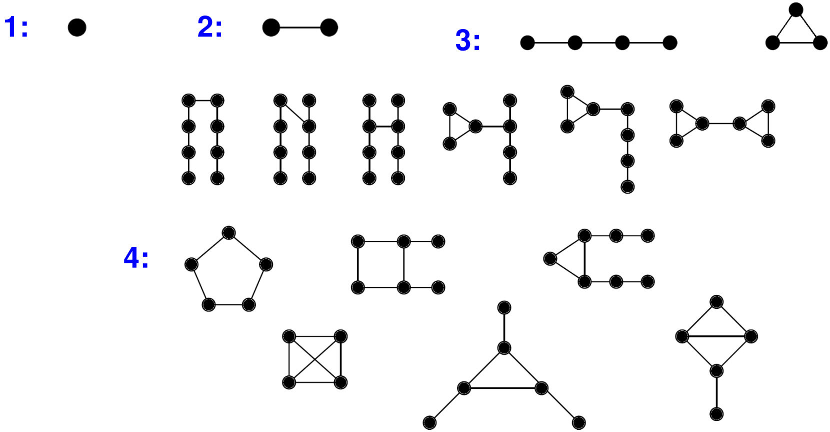


Conjecture 3

If G is k-critical, then G is 1-unique.

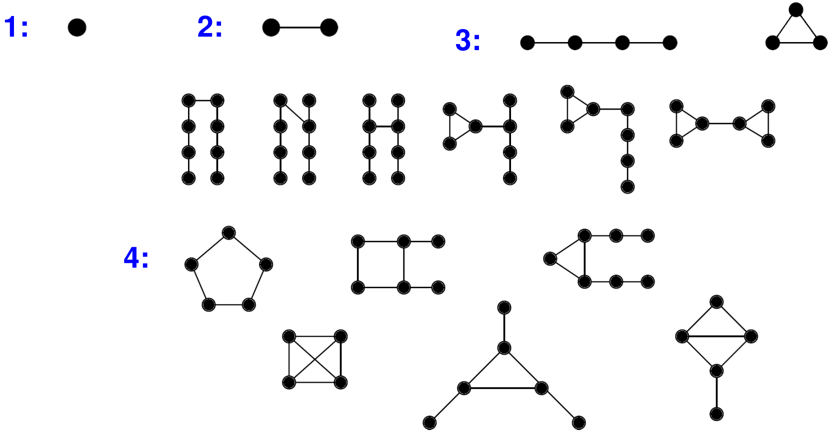
Things to think about



Conjecture 1 [Dvořák–Giannopoulou–Thilikos, '09, '12]

If G is k-critical, then $|V(G)| \leq 2^{k-1}$.

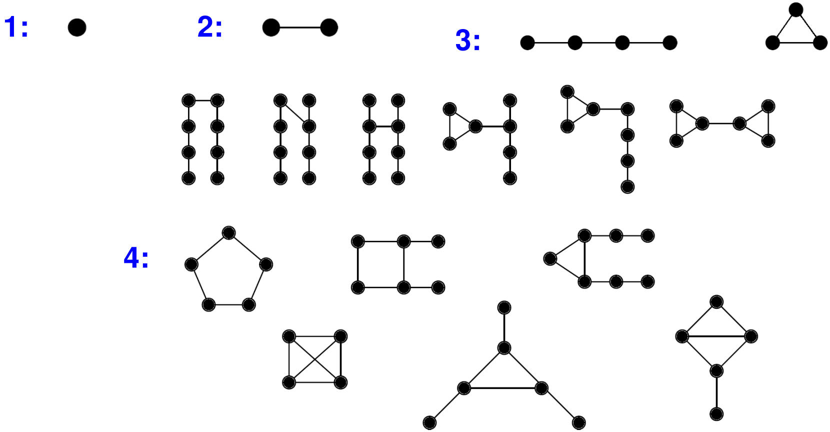
Things to think about



Conjecture 2

If G is k-critical, then $\Delta(G) \le k - 1$.

Things to think about



Thank you!

Tree-depth

(aka (vertex) ranking number, ordered coloring number, ...)

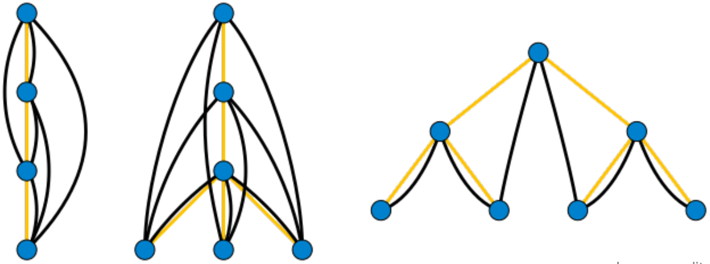


Image credit: wikipedia.org

Equivalently, the smallest height of a tree for which the edges of G all join ancestor-descendant pairs.