Towards spectral characterizations of hereditary graph classes

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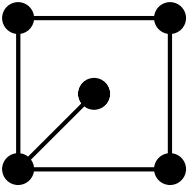
Adjacency spectrum of G



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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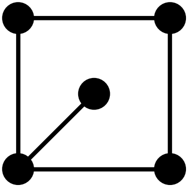

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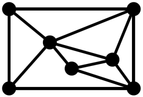
$\text{Spec}(G) =$

$$\left\{ -\sqrt{\frac{1}{2}\left(5+\sqrt{17}\right)}, -\sqrt{\frac{1}{2}\left(5-\sqrt{17}\right)}, 0, \sqrt{\frac{1}{2}\left(5-\sqrt{17}\right)}, \sqrt{\frac{1}{2}\left(5+\sqrt{17}\right)} \right\}$$

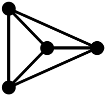
Hereditary graph classes

A graph class is hereditary if it is closed under taking induced subgraphs ($\Leftrightarrow$ under vertex deletions).

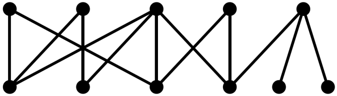
- planar graphs



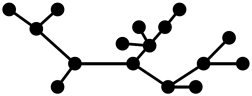
- complete graphs



- bipartite graphs



- forests

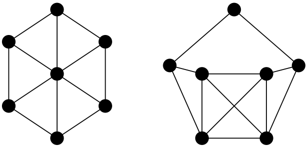


Which hereditary classes are determined by their spectra?

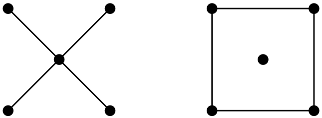
Cospectral graphs

A class $\mathcal{C}$ has no spectral characterization if some cospectral pair has one element in $\mathcal{C}$ and the other not in $\mathcal{C}$.

planar?



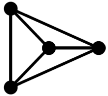
forests?

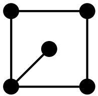


Which hereditary classes are determined by their spectra?

Determined by their spectra

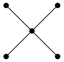
Theorem. K_n is the only graph having spectrum $\{n-1, (-1)^{n-1}\}$, so complete graphs have a spectral characterization.

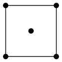




Theorem. G is bipartite $\Leftrightarrow$ Spec(G) is symmetric about 0.

$$\left\{-\sqrt{\frac{1}{2}\left(5+\sqrt{17}\right)}, -\sqrt{\frac{1}{2}\left(5-\sqrt{17}\right)}, 0, \sqrt{\frac{1}{2}\left(5-\sqrt{17}\right)}, \sqrt{\frac{1}{2}\left(5+\sqrt{17}\right)}\right\}$$

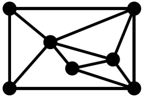




Which hereditary classes are determined by their spectra?

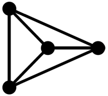
Forbidden subgraphs

- planar graphs



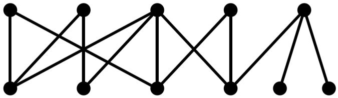
$\{K_5^*, K_{3,3}^*\}$-free

- complete graphs



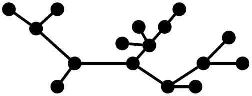
$\{2K_1\}$-free

- bipartite graphs



$\{C_{2n+1}\}$-free

- forests



$\{C_n\}$-free

Call a set $\mathcal{F}$ of graphs **spectrum-forcing** (SF) if the $\mathcal{F}$-free graphs have a spectral characterization. **Which $\mathcal{F}$ are?**

SF sets from closed walks

Theorem. $\sum_{i=1}^{n} \lambda_i^k$ is the number of closed walks of length k.

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Cor. $\{K_2\}$ and $\{K_3\}$ are SF.

$$\sum_{i=1}^{n} \lambda_i^k = \sum_{H \in \mathscr{W}_k} \left(\begin{array}{c} \text{number of} \\ \text{spanning closed} \\ k\text{-walks in } H \end{array} \right) \left(\begin{array}{c} \text{number of} \\ \text{induced copies} \\ \text{of } H \text{ in } G \end{array} \right),$$

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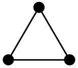

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$$\mathscr{W}_4 = \{K_2, P_3, K_3, C_4, \text{diamond}, K_4\}$$

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$$\sum \lambda_i^2 = 2(\#K_2) \qquad \sum \lambda_i^3 = 6(\#K_3)$$

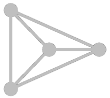
$$\sum \lambda_i^4 = 2(\#K_2) + 4(\#P_3) + 12(\#K_3) + 8(\#C_4) + 8(\#\text{diamond}) + 24(\#K_4)$$

Theorem. The set $\{P_3, K_4\}$ is SF.

What other sets are?

SF sets from DS families

Theorem. K_n is the only graph having spectrum $\{n-1, (-1)^{n-1}\}$, so complete graphs have a spectral characterization.



A graph is DS (determined by its spectrum) if it is the unique graph having its spectrum (i.e., it belongs to no nontrivial cospectral pair).

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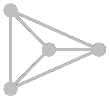

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Observation. If every $\mathcal{F}$-free graph is DS, then $\mathcal{F}$ is SF.

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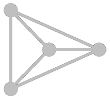

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Theorem. The set $\{P_3\}$ is SF.

Proof. Graph G is a disjoint union of cliques iff G is P_3-free; the graphs $K_{n_1} + \cdots + K_{n_t}$ are all DS. $\square$

SF sets from DS families

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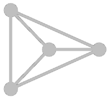

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For which $\mathcal{F}$ are the $\mathcal{F}$-free graphs DS?

SF sets from eigenvalue bounds

Theorem (Interlacing). For any graph G and any vertex v of G, if $H = G - v$ and the spectra of G and H are

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \quad \text{and} \quad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1},$$

respectively, then

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Observation. Graphs with a fixed upper or lower bound on their eigenvalues, or a bound on their spectral radius, form a hereditary class (with a SF set of forbidden subgraphs).

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Theorem (again). The set $\{P_3\}$ is SF.

Proof. Spec(P_3) $= \{0, \pm\sqrt{2}\}$. If G has eigenvalues $\lambda_1 > \cdots > \lambda_n$,

- $\lambda_n > -\sqrt{2}$ implies G is P_3-free;
- G is P_3-free implies G is a disjoint union of complete graphs;
- G a disjoint union of complete graphs implies $\lambda_n = -1$ (or $\lambda_n = 0$)
- $\lambda_n = -1$ (or $\lambda_n = 0$) implies $\lambda_n > -\sqrt{2}$. □

SF sets from eigenvalue bounds

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Which sets $\mathcal{F}$ arise as the minimal graphs having eigenvalues outside some interval in $\mathbb{R}$?

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Theorem (Godsil–McKay, 1982). GM-switching produces cospectral graphs.

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Lots to do

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- Which sets $\mathcal{F}$ are spectrum-forcing?

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Thank you!