Graphs with small Erdős–Gallai differences

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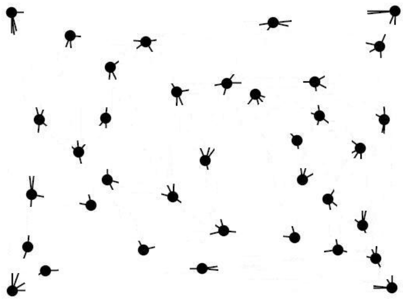
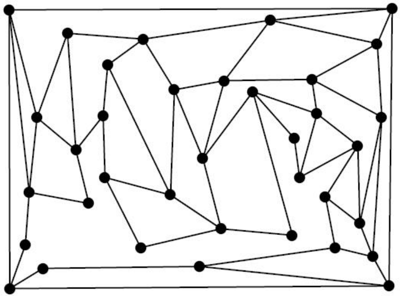
Summer Combo in Vermont
Saint Michael's College • July 21, 2015

Degree sequences, inequalities, and graphs



$$d(G) = (4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,\\ 4,4,4,4,3,3,3,3,3,3,3,3,2,2,2,2,2,2)$$

Degree sequences, inequalities, and graphs



$$d(G) = (4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,\\ 4,4,4,4,3,3,3,3,3,3,3,3,2,2,2,2,2,2)$$

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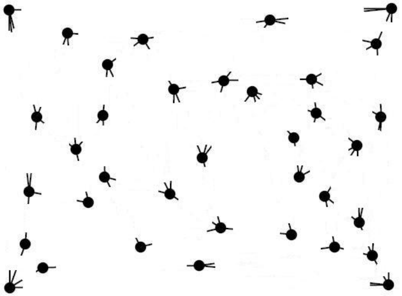

$$d(G) = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, \\ 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2)$$

Degree sequences, inequalities, and graphs

$$(5, 5, 5, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 3, 3,\\ 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2, 1)$$

Erdős–Gallai inequalities

A list $(d_1, \ldots, d_n)$ of nonnegative integers in descending order with even sum is a degree sequence if and only if

$$\sum_{i \leq k} d_i \leq k(k-1) + \sum_{i>k} \min\{k, d_i\}$$

for all $k \leq \max\{i : d_i \geq i-1\}$.

Degree sequences, inequalities, and graphs

$$(5,5,5,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,3,3,$$
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A list $(d_1,\ldots,d_n)$ of nonnegative integers in descending order with even sum is a degree sequence if and only if

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What happens when **equality** holds?

Threshold graphs

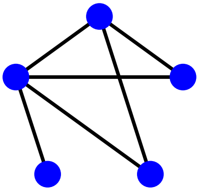
A **threshold sequence** is a list $d = (d_1, \ldots, d_n)$ of nonnegative integers in descending order having even sum and satisfying

$$\sum_{i \leq k} d_i = k(k-1) + \sum_{i>k} \min\{k, d_i\}$$

for all $k \leq \max\{i : d_i \geq i - 1\}$.

A **threshold graph** is a graph having a threshold sequence as its degree sequence.

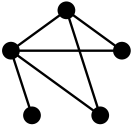
$(4, 3, 2, 2, 1)$



Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

Construction one vertex at a time



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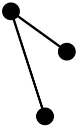

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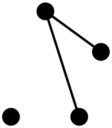

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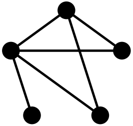

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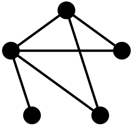
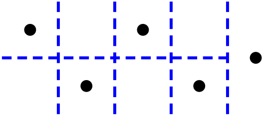


A framework



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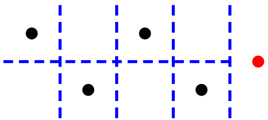

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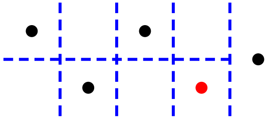

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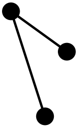

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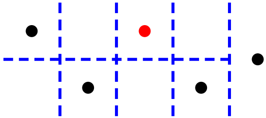

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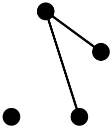

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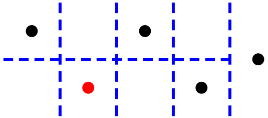

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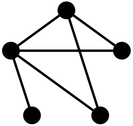

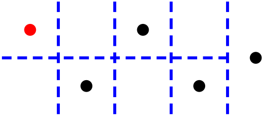

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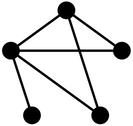

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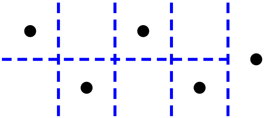


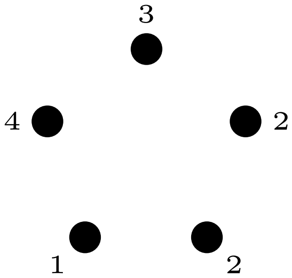
There are exactly 2^{n-1} threshold graphs with n vertices.

Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

Unique realizability (labeled graphs)

$d = (4, 3, 2, 2, 1)$



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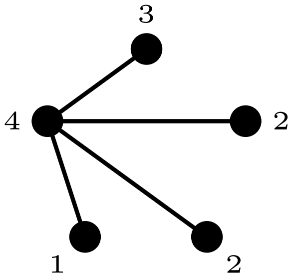

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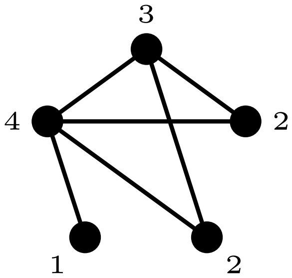

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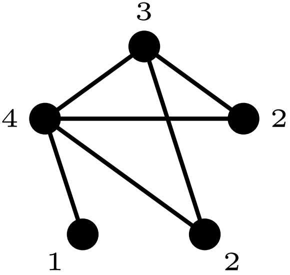

d is a threshold sequence iff it has a unique (labeled) realization.

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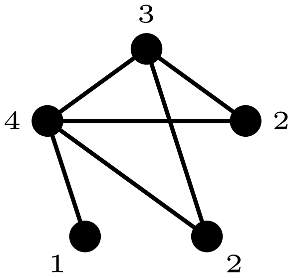


d is a threshold sequence iff it has a unique (labeled) realization.

G is a threshold graph iff it has no induced $2K_2$, C_4, or P_4.

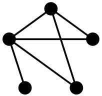
Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

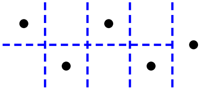
- Equality in the first $m(d)$ Erdős–Gallai inequalities.

  $\sum_{i \le k} d_i = k(k-1) + \sum_{i>k} \min\{k, d_i\}$

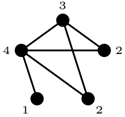
- Iterative construction via dominating/isolated vertices



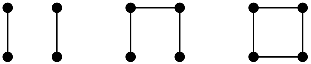
- There are exactly 2^{n-1} threshold graphs on n vertices.



- Unique realization of degree sequence



- $\{2K_2, P_4, C_4\}$-free



- Threshold sequences majorize all other degree sequences

- ...

Erdős–Gallai differences?

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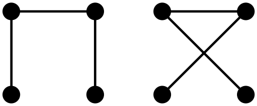
Which adjacency relationships are **forced** by a degree sequence?

$d = (2, 2, 1, 1)$



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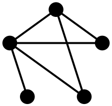

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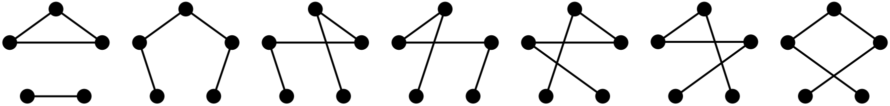

Erdős–Gallai differences

Which adjacency relationships are forced by d?

$$\underbrace{\sum_{i \le k} d_i}_{\text{LHS}_k(d)} \le \underbrace{k(k-1) + \sum_{i>k} \min\{k, d_i\}}_{\text{RHS}_k(d)}$$

$$\Delta_k(d) = \text{RHS}_k(d) - \text{LHS}_k(d)$$

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$$\Delta_k(d) = \text{RHS}_k(d) - \text{LHS}_k(d)$$

Theorem (B,2015+)

Given $1 \le i < j \le n$,
$\{i, j\}$ *is a* ***forced edge*** *iff* $\exists k \in \{1, \dots, n\}$ *such that either* $\Delta_k(d) = 0$, $i \le k < j$, *and* $k \le d_j$; *or* $\Delta_k(d) \le 1$ *and* $j \le k$.

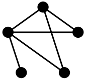
$\{i, j\}$ *is a* ***forced non-edge*** *iff* $\exists k \in \{1, \dots, n\}$ *such that either* $\Delta_k(d) = 0$, $k < i$, *and* $d_j < k \le d_i$; *or* $\Delta_k(d) \le 1$ *and* $d_i < k < i$.

Forcible edges can be determined by examining when $\Delta(k) \le 1$.

Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

- The first $m(d)$ Erdős–Gallai differences equal 0.
- Iterative construction via dominating/isolated vertices



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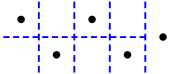
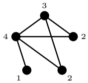
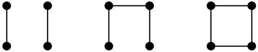


- Unique realization of degree sequence



- $\{2K_2, P_4, C_4\}$-free



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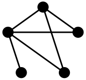

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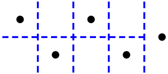

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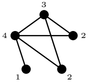

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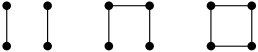

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What if $\Delta_k(d) \leq 1$?

Weakly threshold graphs

A **weakly threshold sequence** is a **graphic** list $d = (d_1, \ldots, d_n)$ of nonnegative integers in descending order having even sum and satisfying $0 \leq \Delta_k(d) \leq 1$ for all $k \leq \max\{i : d_i \geq i - 1\}$.

A **weakly threshold graph** is a graph having a weakly threshold sequence as its degree sequence.

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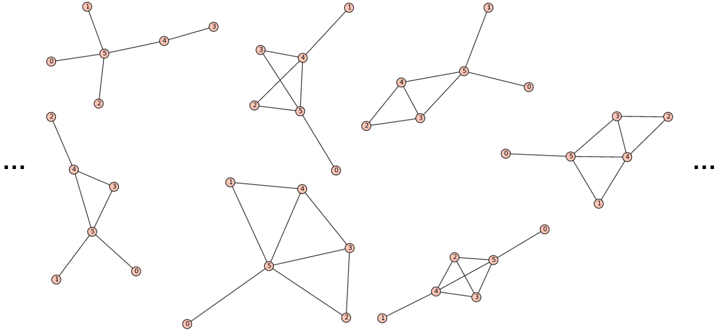
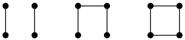


A forbidden subgraph characterization

G is a threshold graph iff G is $\{2K_2, P_4, C_4\}$-free



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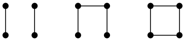
G is a threshold graph iff G is $\{2K_2, P_4, C_4\}$-free 

The class of weakly threshold graphs is **hereditary** (i.e., closed under taking induced subgraphs).

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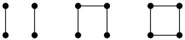

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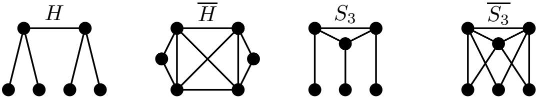
Theorem

A graph G is weakly threshold if and only if it is $\{2K_2, C_4, C_5, H, \overline{H}, S_3, \overline{S_3}\}$-free.



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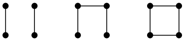

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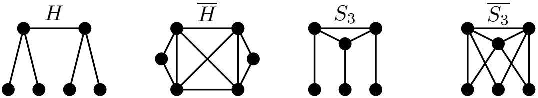

The class is closed under complementation.

Weakly threshold graphs are all **split** graphs.

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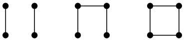

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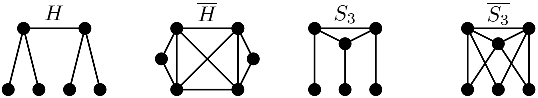
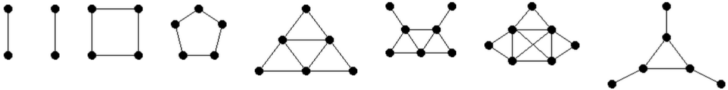
A graph G is weakly threshold if and only if it is $\{2K_2, C_4, C_5, H, \overline{H}, S_3, \overline{S_3}\}$-free.



They form a large subclass of **interval ∩ co-interval.**

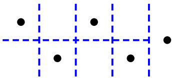
(This class's forbidden induced subgraphs:)



Structural characterization

Threshold iff constructed from K_1 via dominating/ isolated vertices.

Exactly 2^{n-1} threshold graphs on n vertices.



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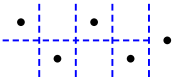

Theorem

A graph is weakly threshold iff it is constructed by "composing" special graphs, where

*a graph is **special** iff it is isomorphic to K_1 or is obtained by starting with P_4 and iteratively adding either **weakly dominating** or **weakly isolated** vertices.*

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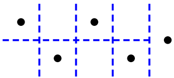

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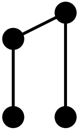

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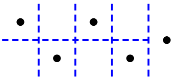

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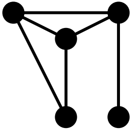

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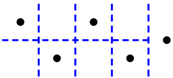

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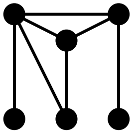

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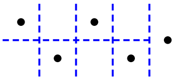

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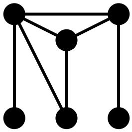

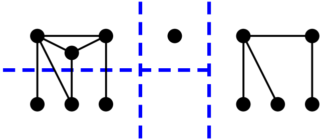

Iterative construction

Threshold iff constructed from K_1 via dominating/isolated vertices.
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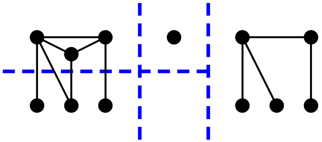

Theorem

A graph is weakly threshold iff it is constructed from K_1 or P_4 by iteratively adding one of

- *a dominating vertex,*
- *a weakly dominating vertex,*
- *an isolated vertex,*
- *a weakly isolated vertex, or*
- *a P_4 with its midpoints dominating all previous vertices.*

Enumeration

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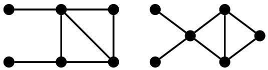
Subtleties in direct counting.

Difference between counting degree sequences / isomorphism classes.



Enumeration

Exactly 2^{n-1} threshold graphs on n vertices.

a_n = number of weakly threshold **sequences** of length n

$$(1,)1, 2, 4, 9, 21, 50, 120, 289, 697, 1682, 4060, \ldots$$

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OEIS.org sequences A024537, A171842

Theorem

$\{a_n\}_n$ satisfies the following recurrences:

- *For all $n \geq 4$,* $a_n = 2a_{n-1} + \sum_{k=0}^{n-4} 2^k a_{n-4-k}$.
- *For all $n \geq 3$,* $a_n = 3a_{n-1} - a_{n-2} - a_{n-3}$.
- *For all $n \geq 4$,* $a_n = 4a_{n-1} - 4a_{n-2} + a_{n-4}$.
- *For all $n \geq 2$,* $a_n = 2a_{n-1} + a_{n-2} - 1$.

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$$a_n = \frac{2 + (1+\sqrt{2})^n + (1-\sqrt{2})^n}{4} \approx \frac{1}{4} \cdot 2.4^n$$

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From OEIS.org:

- Binomial transform of $1, 0, 1, 0, 2, 0, 4, 0, 8, 0, 16, \ldots$
- Number of nonisomorphic n-element interval orders with no 3-element antichain.
- Top left entry of the nth power of $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ or of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.
- Number of $(1, s_1, \ldots, s_{n-1}, 1)$ such that $s_i \in \{1, 2, 3\}$ and $|s_i - s_{i-1}| \leq 1$.
- Partial sums of the Pell numbers prefaced with a 1.
- The number of ways to write an $(n-1)$-bit binary sequence and then give runs of ones weakly incrementing labels starting with 1, e.g., 00110100110222030033300440400555555.
- Lower bound of the order of the set of equivalent resistances of $(n-1)$ equal resistors combined in series and in parallel.

Properties of weakly threshold graphs

(B, 2015+)

- The first $m(d)$ Erdős–Gallai differences equal 0 or 1.
- Iterative construction via (weakly) dominating vertices/(weakly) isolated vertices/half-dominating P_4s.
- There are exactly

$$\frac{2 + (1+\sqrt{2})^n + (1-\sqrt{2})^n}{4}$$

weakly threshold sequences of length n.

- Constrained realizations of degree sequences
- $\{2K_2, C_4, C_5, H, \overline{H}, S_3, \overline{S_3}\}$-free
- Weakly threshold sequences at the top of the majorization poset
- ?

Thank you!

barrus@uri.edu