Realization graphs of degree sequences

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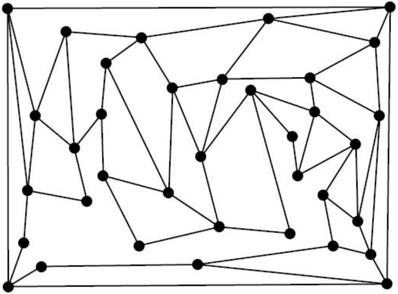
Graphs and degree sequences



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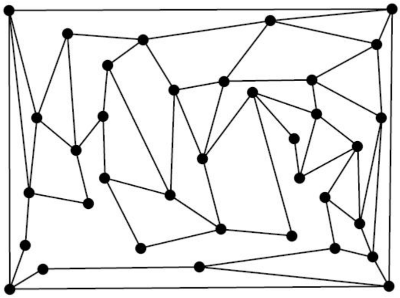

$$d(G) = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2)$$

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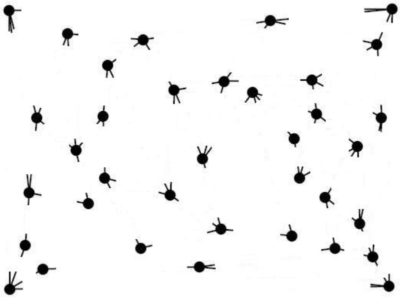

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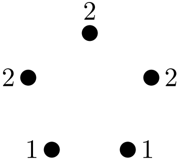
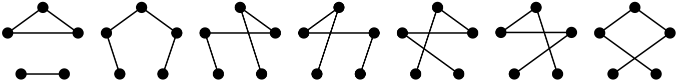
Realizations and 2-switches

$(2, 2, 2, 1, 1)$



The realizations



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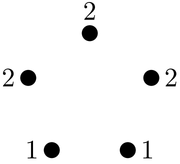

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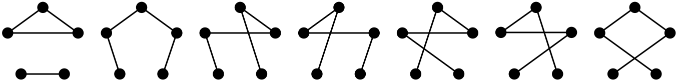
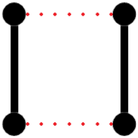
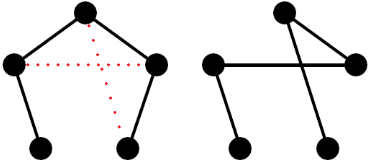


Alternating 4-cycle

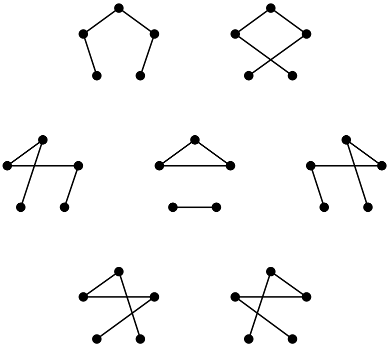


2-switch



The realization graph of d

$d = (2, 2, 2, 1, 1)$



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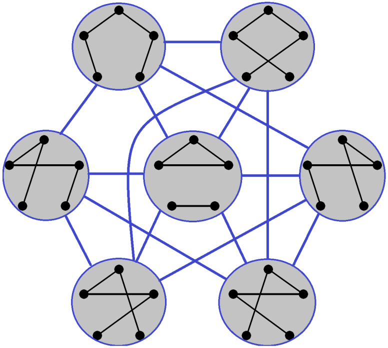

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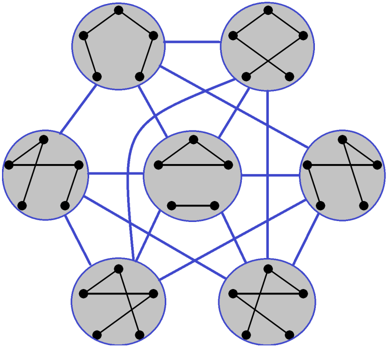

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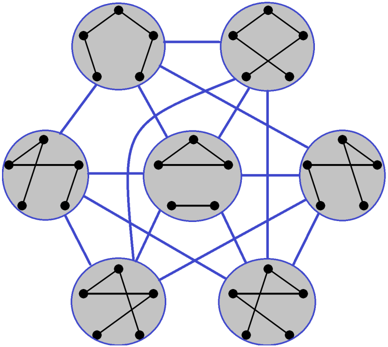

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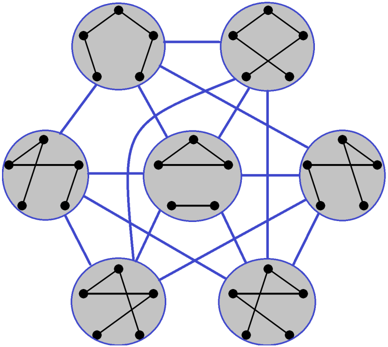

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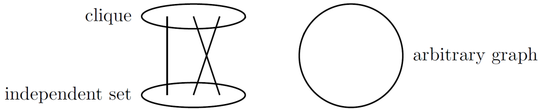
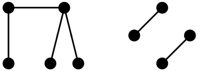
Canonical decomposition

[Tyshkevich, ∼1980, 2000]

Composing a split graph
with a graph:





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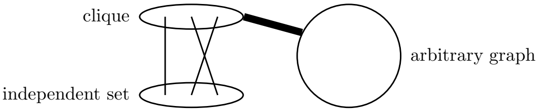

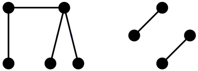

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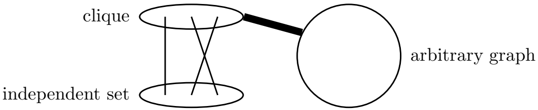

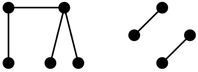

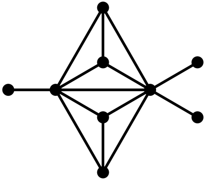

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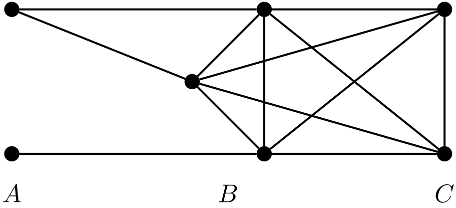

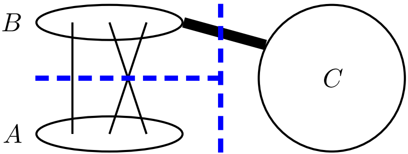

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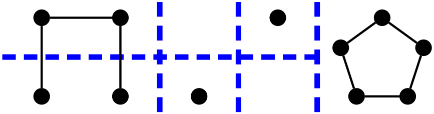

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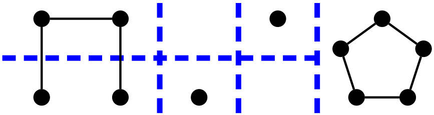


Theorem

Every graph F can be represented as a composition

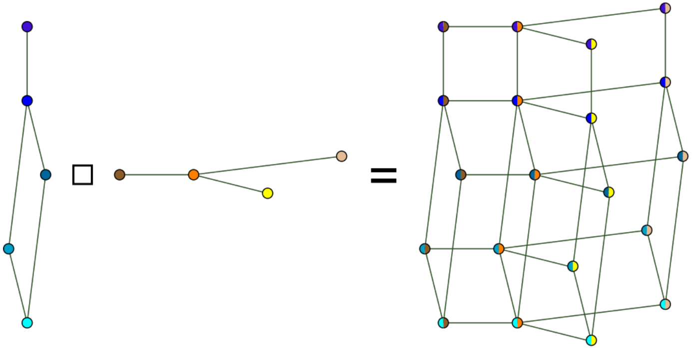
$$F = (G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ F_0$$

of indecomposable components. Here the (G_i, A_i, B_i) are indecomposable splitted graphs and F_0 is an indecomposable graph. This decomposition is unique up to isomorphism of components.

Cartesian products of graphs

$$V(G \square H) = V(G) \times V(H),$$

$$E(G \square H) = \{\text{pairs } (u, v), (u, w) \text{ such that } v \leftrightarrow w \text{ in } H\} \cup \{\text{pairs } (x, y), (z, y) \text{ such that } x \leftrightarrow z \text{ in } G\}$$



Realization graph products

Theorem

If a degree sequence d has a realization F with canonical decomposition

$$F = (G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ F_0,$$

then

$$R(d) = R\left(\deg(G_k)\right) \Box \cdots \Box R\left(\deg(G_1)\right) \Box R\left(\deg(G_0)\right).$$

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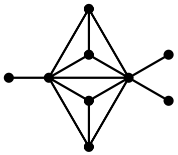

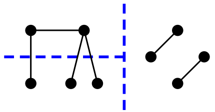

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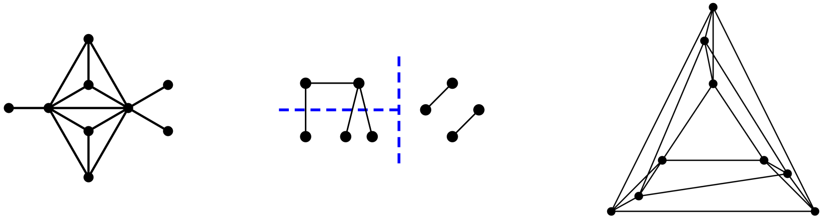

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Theorem

Let d be a degree sequence. The realization graph $R(d)$ is a hypercube if and only if d is the degree sequence of a split P_4-reducible graph.

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$$R(d) = R\left(\deg(G_k)\right) \Box \cdots \Box R\left(\deg(G_1)\right) \Box R\left(\deg(G_0)\right).$$

Corollary

If each of $R(\deg(G_k)), \ldots, R(\deg(G_0))$ is Hamiltonian, then $R(d)$ is Hamiltonian as well.

Induced subgraphs and realization graphs

Theorem

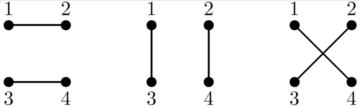
In any infinite list of realization graphs, one of them is an induced subgraph of some other.

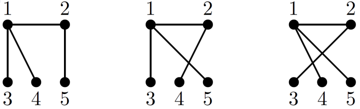
$$R(d_1) \qquad R(d_2) \qquad R(d_3) \qquad \ldots$$

Induced subgraphs and realization graphs

Proposition

For degree sequences d and e, if d has a realization that is an induced subgraph of some realization of e, then $R(d)$ is an induced subgraph of $R(e)$.





Theorem

The following are equivalent for a degree sequence d:

- *$R(d)$ is bipartite;*
- *$R(d)$ is triangle-free;*
- *$R(d)$ is the Cartesian product of transposition graphs and at most one copy of $K_{6,6} - 6K_2$;*
- *d is the degree sequence of a pseudo-split matrogenic graph.*

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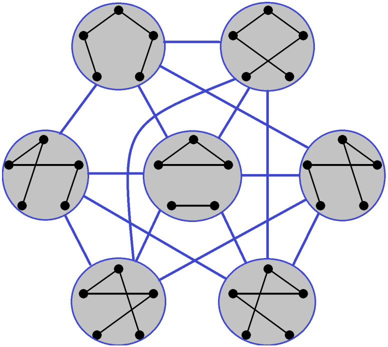

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- Well-quasi-ordering
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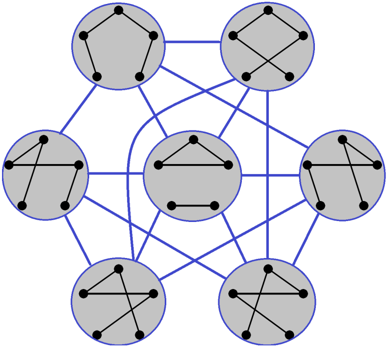

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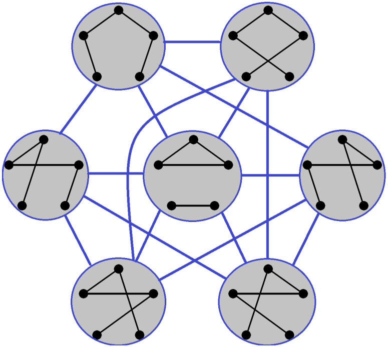

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