# Realization graphs of degree sequences

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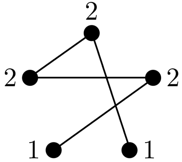
# Graphs, degree sequences, realizations, and 2-switches

$(2, 2, 2, 1, 1)$



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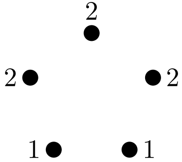

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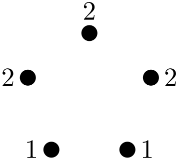
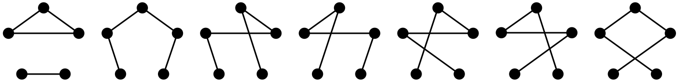



The realizations



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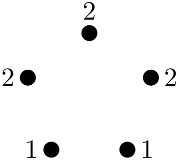


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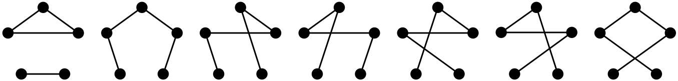
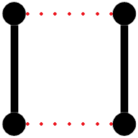
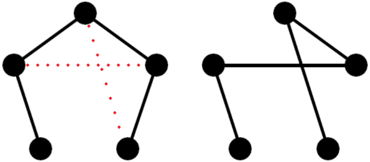


Alternating 4-cycle



2-switch



# The realization graph of $d$

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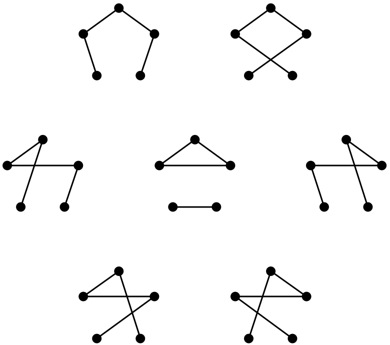

$$V(R(d)) = \{\text{realizations of } d\},$$
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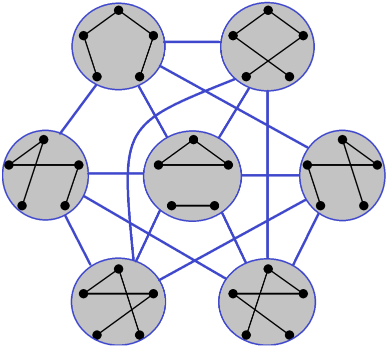

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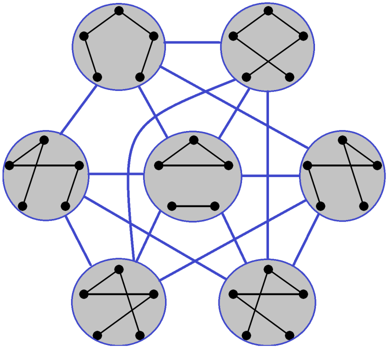

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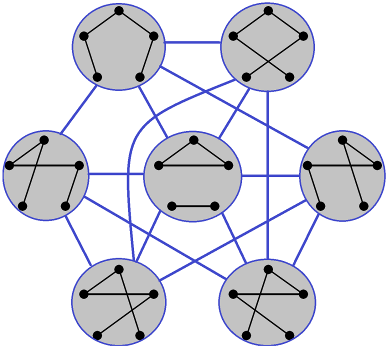

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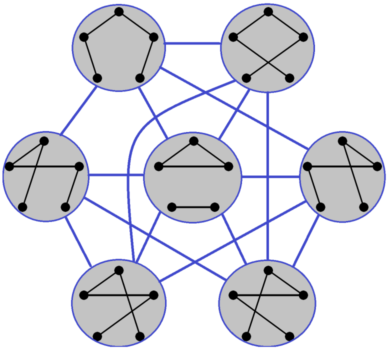

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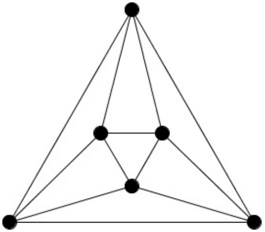
- Bounds on distances.
- Various conditions on $d$ imply $R(d)$ is Hamiltonian.

# Connections in realization graphs

(B, 2016+)

In a realization graph $G$,

$R((2,1,1,1,1))$



**Proposition**

Every path of length 2 belongs to a triangle or 4-cycle.

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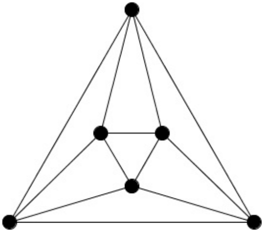

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**Corollary**

*$G$ is $K_1$ or $K_2$, or $G$ has girth 3 or 4.*

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*$G$ is $K_1$ or $K_2$, or $G$ is 2-connected.*

# Induced subgraphs and realization graphs

(B, 2016)

## Proposition

For degree sequences $d$ and $e$, if $d$ has a realization that is an induced subgraph of some realization of $e$, then $R(d)$ is an induced subgraph of $R(e)$.

## Theorem

Realization graphs form a WQO under the induced subgraph order. In other words, in any infinite list of realization graphs, one of them is an induced subgraph of some other.

$$R(d_1) \qquad R(d_2) \qquad R(d_3) \qquad \ldots$$

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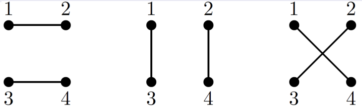
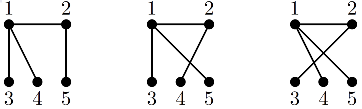



## Theorem

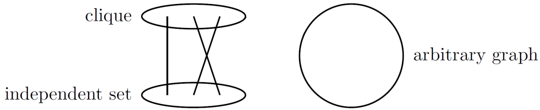
*The following are equivalent for a degree sequence $d$:*

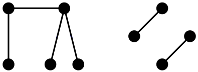
- *$R(d)$ is bipartite;*
- *$R(d)$ is triangle-free;*
- *$R(d)$ is the Cartesian product of transposition graphs and at most one copy of $K_{6,6} - 6K_2$;*
- *$d$ is the degree sequence of a pseudo-split matrogenic graph.*

# Canonical decomposition

[Tyshkevich, ∼1980, 2000]

Composing a splitted graph with a graph:



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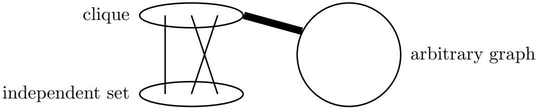


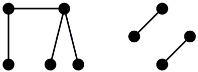

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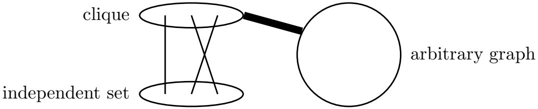


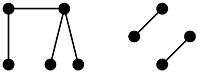

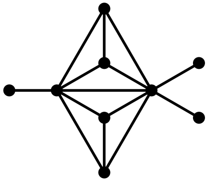

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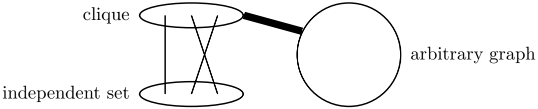


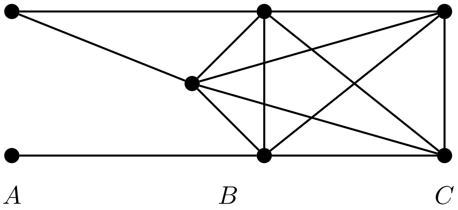

Composing degree sequences:

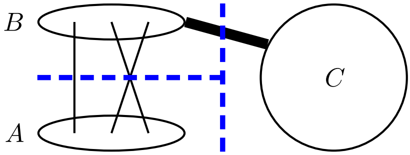
$$(3, 2; 1, 1, 1) \circ (1, 1, 1, 1) \quad = \quad (7, 6, 3, 3, 3, 3, 1, 1, 1)$$

# Canonical decomposition

[Tyshkevich, ∼1980, 2000]

Decomposing a graph:




Decomposing a degree sequence:

$$(5,5,5,4,4,2,1) \qquad = \qquad (3,3,3;2,1) \circ (1,1)$$

# Canonical decomposition

[Tyshkevich, ∼1980, 2000]

Decomposing a degree sequence:

$$(9,9,7,5,5,5,5,5,2,1,1) \quad = \quad (2,2;1,1)\circ(;0)\circ(0;)\circ(2,2,2,2,2)$$

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## Theorem

*Every degree sequence $d$ can be uniquely expressed (modulo some minor details) as a composition of indecomposable degree sequences, i.e.,*

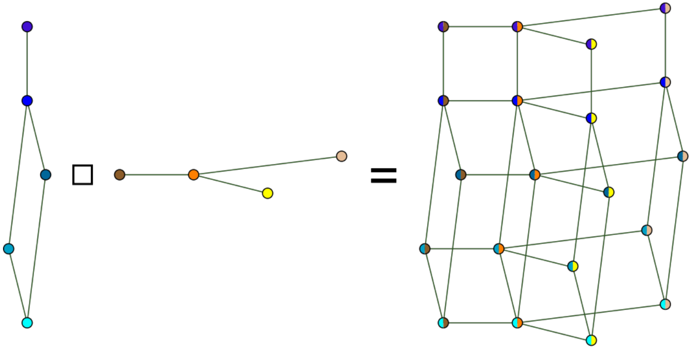
$$d = (\pi_k^C;\pi_k^I)\circ\cdots\circ(\pi_1^C;\pi_1^I)\circ\pi_0$$

*for indecomposable splitted sequences $(\pi_i^C;\pi_i^I)$ and indecomposable sequence $\pi_0$.*

# Cartesian products of graphs

$$V(G \square H) = V(G) \times V(H),$$

$$E(G \square H) = \{\text{pairs } (u, v), (u, w) \text{ such that } v \leftrightarrow w \text{ in } H\} \cup \{\text{pairs } (x, y), (z, y) \text{ such that } x \leftrightarrow z \text{ in } G\}$$



# Realization graph products

(B, 2016)

**Theorem**

*If a degree sequence $d$ has canonical decomposition*

$$d = (\pi_k^C; \pi_k^I) \circ \cdots \circ (\pi_1^C; \pi_1^I) \circ \pi_0,$$

*then*

$$R(d) = R(\pi_k) \square \cdots \square R(\pi_1) \square R(\pi_0).$$

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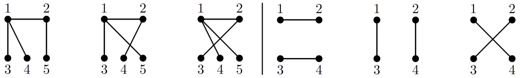
$$= (3, 2; 1, 1, 1) \circ (1, 1, 1, 1)$$



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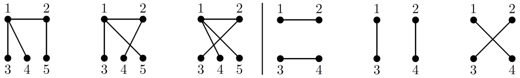


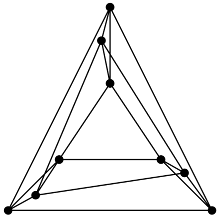

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$$R(d) = R(\pi_k) \,\square \cdots \square\, R(\pi_1) \,\square\, R(\pi_0) \,.$$

**Theorem**

*Let $d$ be a degree sequence. The realization graph $R(d)$ is a hypercube if and only if $d$ is the degree sequence of a split $P_4$-reducible graph.*

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*then*

$$R(d) = R(\pi_k) \square \cdots \square R(\pi_1) \square R(\pi_0).$$

## Corollary

*If each of $R(\pi_k), \ldots, R(\pi_0)$ is Hamiltonian, then $R(d)$ is Hamiltonian as well.*

# The realization graph of $d$

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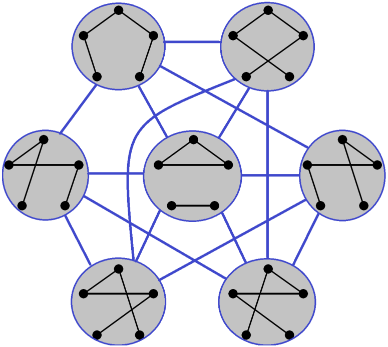

$$V(R(d)) = \{\text{realizations of } d\},$$
$$E(R(d)) = \{\text{pairs joined by a 2-switch}\}$$

**Known:**

- $R(d)$ **2-connected** for virtually all $d$.
- Bounds on distances.
- **Well-quasi-ordered...**
- **Special types of $R(d)$.**
- **Cartesian product decomposition...**
- **Various conditions on $d$ imply $R(d)$ is Hamiltonian.**

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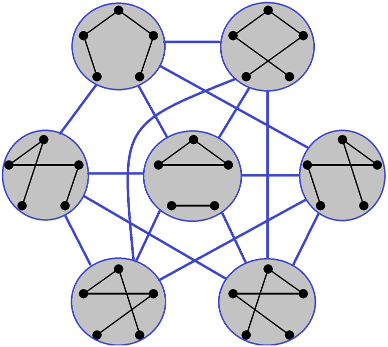

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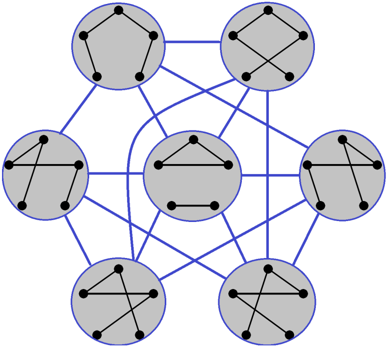

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