Graph Classes with Near–Equality of Independence Numbers and Havel–Hakimi Residues

Michael D. Barrus

Department of Mathematics
University of Rhode Island

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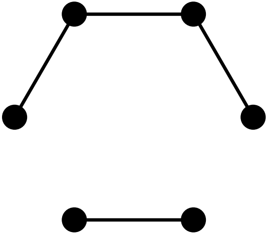
Joint work in part with
Grant Molnar (Brigham Young University)

The Havel–Hakimi Algorithm

Delete, reduce, reorder



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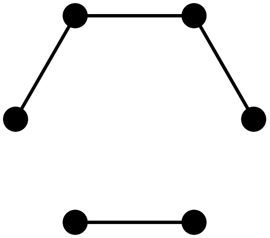

$$d = (\textcolor{red}{\mathbf{2}}, \underline{2}, \underline{1}, 1, 1, 1)$$
$$d^1 = \quad (1, 0, 1, 1, 1)$$

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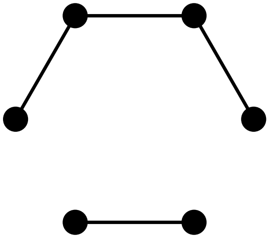

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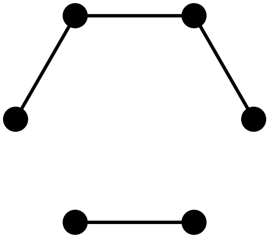

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Theorem (V. Havel, 1995; S.L. Hakimi, 1962)

d is the degree sequence of a simple graph if and only if d^1 is.

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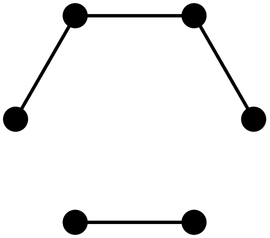

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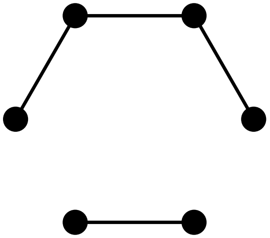

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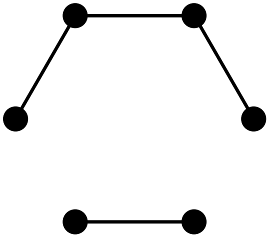

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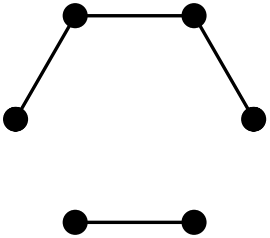

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The **residue** $R(d)$ or $R(G)$ is the number of zeroes remaining at the end.

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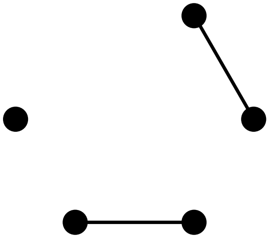

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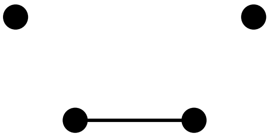

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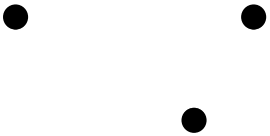

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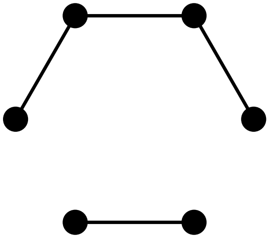

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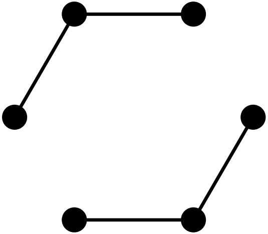

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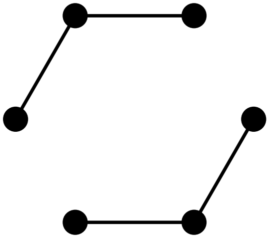

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Theorem (Favaron–Mahéo–Saclé, 1991)

For all graphs G, $R(G) \leq \alpha(G)$.

Big Questions

- **How tight is the** $R(G) \leq \alpha(G)$ **bound?**

- **For which graphs** G **does** $R(G) = \alpha(G)$**?**

How tight is the $R(G) \leq \alpha(G)$ bound?

- One of the tightest known lower bounds
  - rigorously: (Favaron et al., '91) outperforms Brook's, Turan's, Hansen's, Caro–Wei's
  - anecdotally: (Nelson, '01); (Willis, '11) outperforms Wilf's bound?
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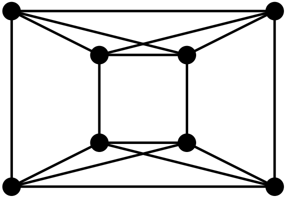

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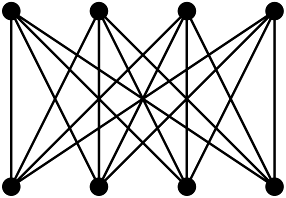

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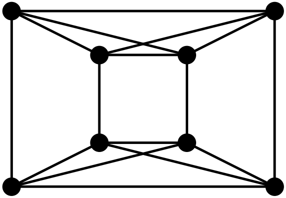

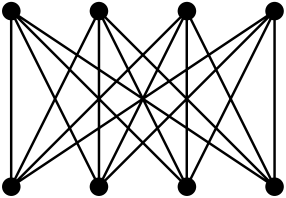

$$R(d) \leq \alpha_{\min}(d) = \alpha(G) \leq \cdots \leq \alpha(G')$$

How tight is the $R(G) \leq \alpha_{\min}(d)$ bound?

- The inequality may be proper:

$$d = (4,4,4,4,4,4,4,4,4) \qquad R(d) = 2 \qquad \alpha_{\min}(d) = 3$$

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- The inequality may be proper:

$$d = (4,4,4,4,4,4,4,4,4) \qquad R(d) = 2 \qquad \alpha_{\min}(d) = 3$$

- **Theorem** (Nelson–Radcliffe, 2004)
  If d is **semi-regular**, then

$$R(d) \leq \alpha_{\min}(d) \leq R(d) + 1,$$

  and we know which d are which.

How tight is the $R(G) \leq \alpha(G)$ bound?

An idea:

$$R(d) \leq \alpha_{\text{min}}(d) = \underbrace{\alpha(G) \leq \cdots \leq \alpha(G')}_{\text{What if this can't be large?}}$$

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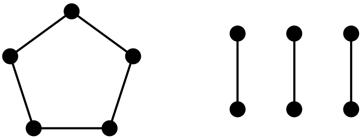
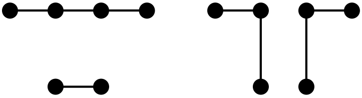
A **unigraph** is a graph that is the **unique realization** (up to isomorphism) of its degree sequence.



UNIGRAPHS



NOT UNIGRAPHS

How tight is the $R(G) \leq \alpha(G)$ bound?

Theorem (B, 2012)

*If G is a **unigraph**, then*

$$R(G) \leq \alpha(G) \leq R(G) + 1,$$

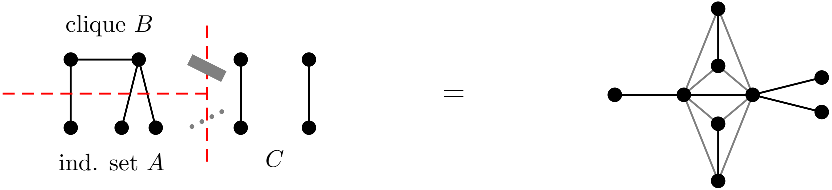
and we know which G are which.

(In fact, $R < \alpha$ in only one simple family of counterexamples.)

Key ideas of the proof

If G is a **unigraph**, then $R(G) \leq \alpha(G) \leq R(G) + 1$.

- Tyshkevich ('00) studied graph compositions of the form



She characterized unigraphs in terms of indecomposable components.

Lemma (B, 2012)

For a graph $G = (G_1, A, B) \circ G_0$, *both*

$$\alpha(G) = |A| + \alpha(G_0) \qquad \text{and} \qquad R(G) = |A| + R(G_0).$$

Big Questions

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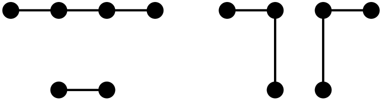
- **For which graphs** G **does** $R(G) = \alpha(G)$**?**

Mimicking vertex deletions in hopes of $R(G) = \alpha(G)$

Joint with Grant Molnar (BYU)



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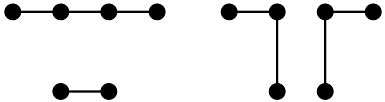

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A graph has the **strong Havel–Hakimi property** if in **every** induced subgraph **every** vertex of maximum degree has neighbors with as high of degrees as possible.

Let $\mathcal{S}$ be the class of all such graphs.

Results (B, Molnar, 2015+)

$\mathcal{S} = \{$graphs with strong Havel–Hakimi property$\}$

- All graphs in $\mathcal{S}$ can be constructed via the natural "reverse Havel–Hakimi process."

$$(0,0,0) \to (1,1,0,0) \to (1,1,1,1,0) \to (2,2,1,1,1,1)$$

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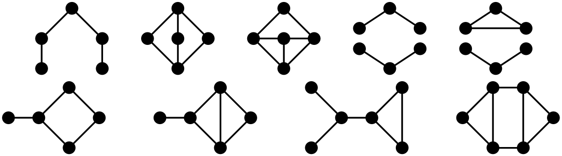

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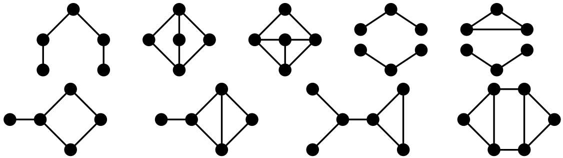

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Hereditary classes of graphs for which $R = \alpha$

(B, 2012, 2013; B, Molnar, 2015+)

- **Split graphs**
- **Hereditary unigraphs**
- $\mathcal{S}$ ($\supset$ matrogenic graphs, matroidal graphs, threshold graphs)

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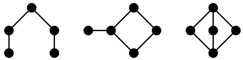
Subgraphs	3	1	1			



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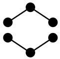

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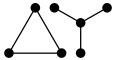

No strong patterns

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No strong patterns, or end in sight

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No strong patterns, or end in sight (??)

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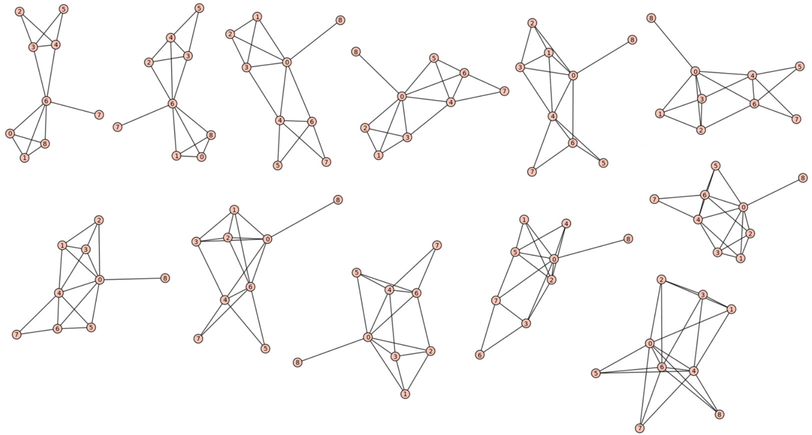
Some forbidden subgraphs for $\mathcal{H}$ with 9 vertices



Thank you!

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