Graphs with low Erdős–Gallai differences

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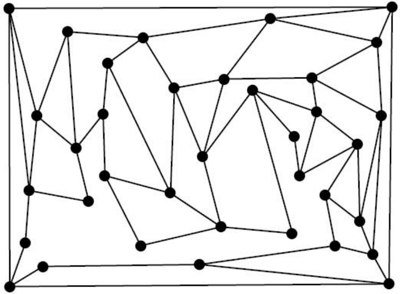
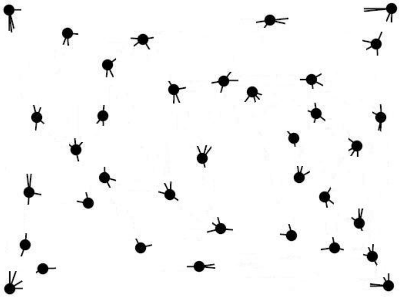
Degree sequences, inequalities, and graphs



$$d(G) = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4,$$
$$4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2)$$

Degree sequences, inequalities, and graphs



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Degree sequences, inequalities, and graphs

$$(5, 5, 5, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 3, 3,$$
$$3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2, 1)$$

Degree sequences, inequalities, and graphs

$$(5, 5, 5, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 3, 3,\\ 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2, 1)$$

Erdős–Gallai inequalities (1960)

A list $(d_1, \ldots, d_n)$ of nonnegative integers in descending order with even sum is a degree sequence if and only if

$$\sum_{i \leq k} d_i \leq k(k-1) + \sum_{i > k} \min\{k, d_i\}$$

for all $k \leq \max\{i : d_i \geq i - 1\}$.

Degree sequences, inequalities, and graphs

$$(5,5,5,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,3,3,\\ 3,3,3,3,3,3,3,3,3,3,3,3,2,2,2,2,2,2,1)$$

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for all $k \leq \max\{i : d_i \geq i-1\}$.

What happens when **equality** holds?

Threshold graphs

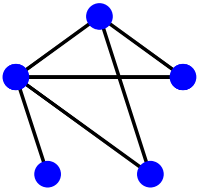
A **threshold sequence** is a list $d = (d_1, \ldots, d_n)$ of nonnegative integers in descending order having even sum and satisfying

$$\sum_{i \le k} d_i = k(k-1) + \sum_{i > k} \min\{k, d_i\}$$

for all $k \le \max\{i : d_i \ge i - 1\}$.

A **threshold graph** is a graph having a threshold sequence as its degree sequence.

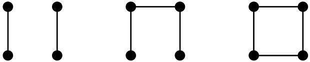
$(4, 3, 2, 2, 1)$



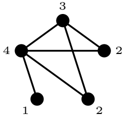
Properties of threshold graphs
(Chvátal, Hammer, others, 1973+)

- Equality in the first $m(d)$ Erdős–Gallai inequalities.
  $\sum_{i\le k} d_i = k(k-1) + \sum_{i>k} \min\{k, d_i\}$
- Iterative construction via dominating/isolated vertices

- There are exactly 2^{n-1} threshold graphs on n vertices.
- $\{2K_2, P_4, C_4\}$-free



- Unique realization of degree sequence



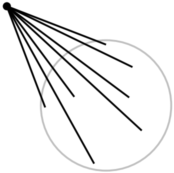
- Threshold sequences majorize all other degree sequences
- ...

Graph building

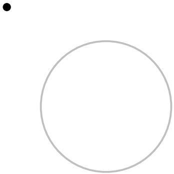
Graph building

Options for adding

Dominating vertex



Isolated vertex



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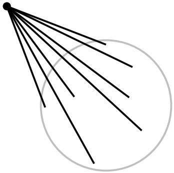

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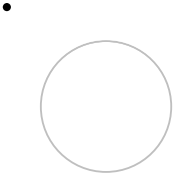

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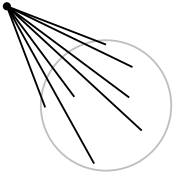

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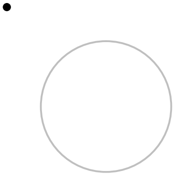

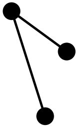

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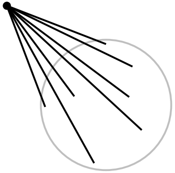

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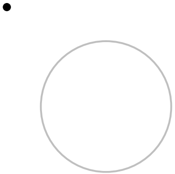

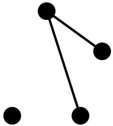

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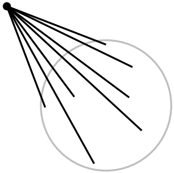

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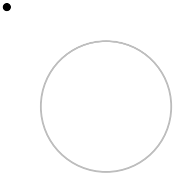

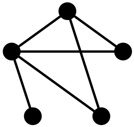

Graph building

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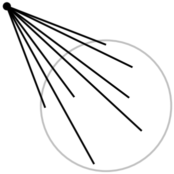

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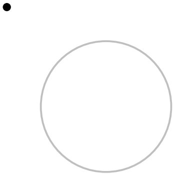

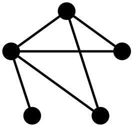

G is a threshold graph if and only if *G* can be constructed from a single vertex via these operations.

Graph building

Options for adding

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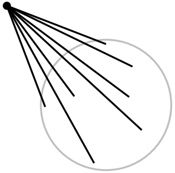

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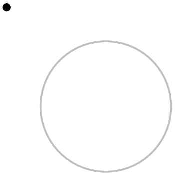
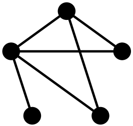


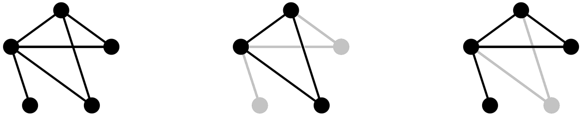


G is a threshold graph if and only if G can be constructed from a single vertex via these operations.

Consequently, up to isomorphism there are exactly 2^{n-1} threshold graphs on n vertices.

A forbidden subgraph characterization

Induced subgraph: a subgraph obtained by deleting vertices and their incident edges



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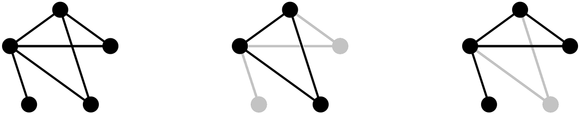
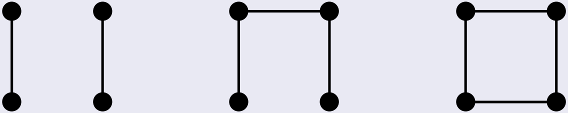


Theorem (Chvátal–Hammer, 1973)

Any induced subgraph of a threshold graph is a threshold graph. In fact, G is a threshold graph iff G has no induced subgraph isomorphic to one of the following:



(We say that G is $\{2K_2, P_4, C_4\}$-free.)

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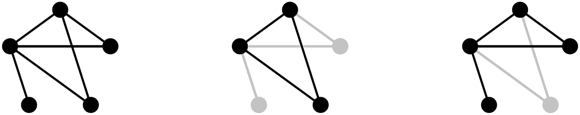

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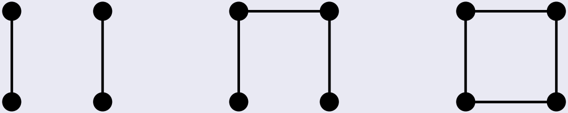

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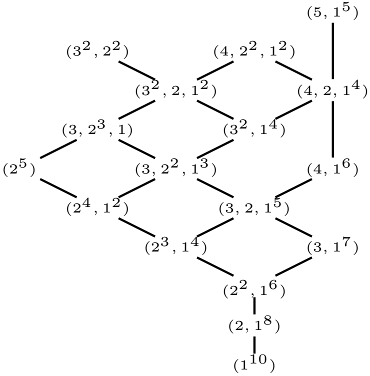
Consequently, for every threshold sequence there is only one threshold graph.

Threshold sequences and majorization



Theorem (Ruch–Gutman, 1979; Peled–Srinivasan, 1989)

d is a threshold sequence if and only if d is a maximal element in the poset of all degree sequences with the same sum, ordered by majorization.

Properties of threshold graphs

(Chvátal, Hammer, others, 1973+)

- Equality in the first $m(d)$ Erdős–Gallai inequalities.
  $\sum_{i \leq k} d_i = k(k-1) + \sum_{i>k} \min\{k, d_i\}$
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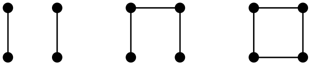
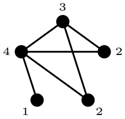


- Unique realization of degree sequence



- Threshold sequences majorize all other degree sequences
- ...

Weakly threshold sequences and graphs

(B, 2017+)

- Equality or a difference of 1 in each of the first $m(d)$ Erdős–Gallai inequalities.

  $k(k-1) + \sum_{i>k} \min\{k, d_i\} - \sum_{i \le k} d_i \le 1$

  Call these **weakly threshold sequences**; call the associated graphs **weakly threshold graphs**.

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- Iterative construction?

- How many weakly threshold sequences/graphs on n vertices?

- Forbidden subgraph characterization?

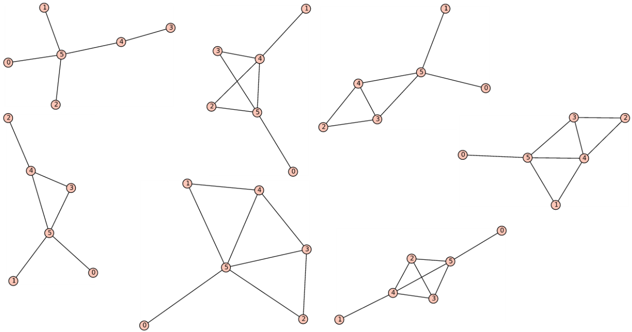
- Unique realizations of degree sequences?

- Majorization result?

- ...?...

Non-threshold, weakly threshold graphs

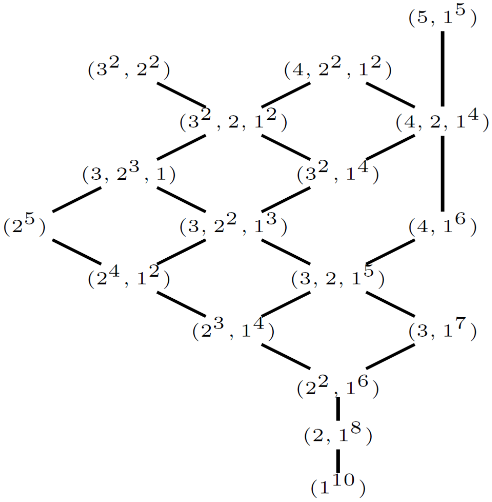
...



...

Near the threshold

Threshold sequences majorize all other degree sequences.



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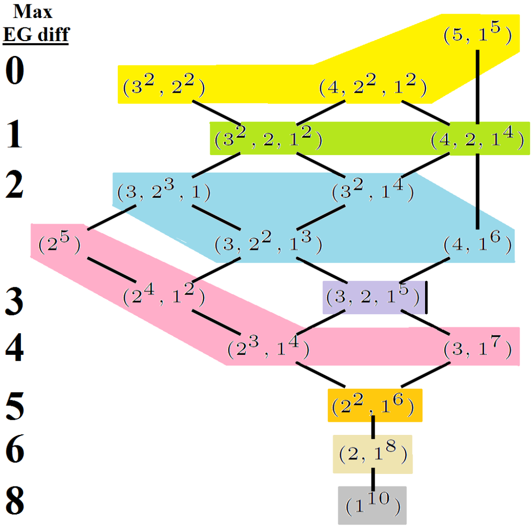

WT sequences (and all diff $\leq b$) are upwards-closed, continue to majorize.

Weakly threshold sequences and graphs

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- ...?...

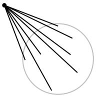
Iterative construction

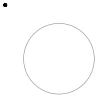
Theorem

G is a threshold graph if and only if G can be constructed by beginning with a single vertex and iteratively adding

- *a dominating vertex, or*
- *an isolated vertex*

Iterative construction

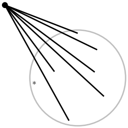


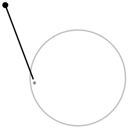


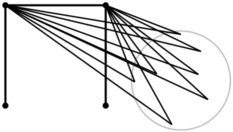
Theorem

*G is a **weakly** threshold graph if and only if G can be constructed by beginning with a single vertex **or** P_4 and iteratively adding*

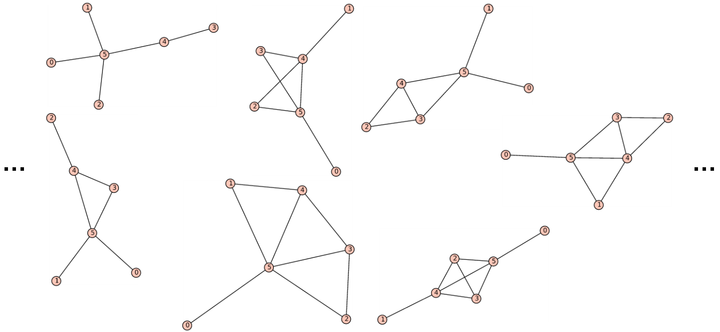
- *a dominating vertex, or*
- *an isolated vertex**, or***
- ***a weakly dominating vertex, or***
- ***a weakly isolated vertex, or***
- ***a semi-joined** P_4**.***







Non-threshold, weakly threshold graphs



Weakly threshold sequences and graphs

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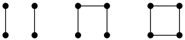
  Call these **weakly threshold sequences**; call the associated graphs **weakly threshold graphs**.

- **Iterative construction** ☑

- How many weakly threshold sequences/graphs on n vertices?
- Forbidden subgraph characterization?
- Unique realizations of degree sequences?
- **Majorization result** ☑
- ...?...

A forbidden subgraph characterization

G is a threshold graph iff G is $\{2K_2, P_4, C_4\}$-free



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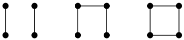
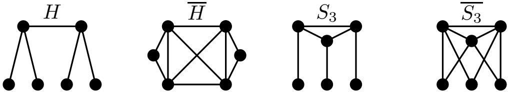


Theorem

The class of weakly threshold graphs is closed under taking induced subgraphs.
In fact, a graph G is weakly threshold if and only if it is $\{2K_2, C_4, C_5, H, \overline{H}, S_3, \overline{S_3}\}$-free.



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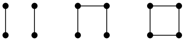

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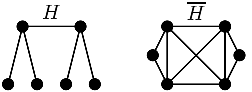
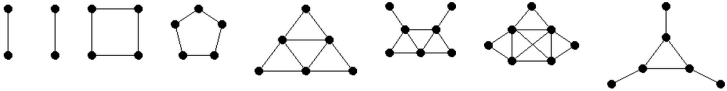


Weakly threshold graphs form a large subclass of **interval** $\cap$ **co-interval**.

(The latter class's forbidden induced subgraphs:)



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- **Iterative construction** ☑

- How many weakly threshold sequences/graphs on n vertices?

- **Forbidden subgraph characterization** ☑

- Unique realizations of degree sequences?

- **Majorization result** ☑

- ...?...

Enumeration: more subtle

Threshold iff constructed from • via dominating/isolated vertices; therefore, exactly 2^{n-1} threshold graphs on n vertices.

A graph is weakly threshold iff it is constructed from a single vertex or P_4 by iteratively adding one of ...

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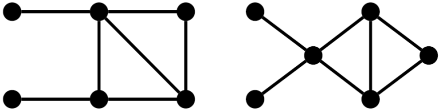
One wrinkle (of many): there is a difference between counting weakly threshold sequences / weakly threshold graphs (isomorphism classes).



Unlike threshold sequences, some weakly threshold sequences have multiple realizations!

Enumeration: sequences

a_n = number of weakly threshold **sequences** of length n

Proposition: For all $n \geq 4$, $a_n = 4a_{n-1} - 4a_{n-2} + a_{n-4}$.

$$(1,)1, 2, 4, 9, 21, 50, 120, 289, 697, 1682, 4060, \ldots$$

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It's in OEIS.org! Sequences A024537, A171842

- Binomial transform of 1, 0, 1, 0, 2, 0, 4, 0, 8, 0, 16, . . .
- Number of nonisomorphic n-element interval orders with no 3-element antichain.
- Top left entry of the nth power of $\begin{bmatrix}1 & 1 & 0\\1 & 1 & 1\\0 & 1 & 1\end{bmatrix}$ or of $\begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & 1 & 1\end{bmatrix}$.
- Number of $(1, s_1, \ldots, s_{n-1}, 1)$ such that $s_i \in \{1, 2, 3\}$ and $|s_i - s_{i-1}| \leq 1$.
- Partial sums of the Pell numbers prefaced with a 1.
- The number of ways to write an $(n-1)$-bit binary sequence and then give runs of ones weakly incrementing labels starting with 1, e.g., 00110100110222030033300440400555555.
- Lower bound of the order of the set of equivalent resistances of $(n-1)$ equal resistors combined in series and in parallel.

Enumeration: graphs

b_n = number of weakly threshold **graphs** with n vertices

Theorem

The generating function for (b_n) *is given by*

$$\sum_{n=0}^{\infty} b_n x^n = \frac{x - 2x^2 - x^3 - x^5}{1 - 4x + 3x^2 + x^3 + x^5}.$$

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$$\begin{aligned}
b_n = {} & c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \\
& + c_3 \left(\frac{6 - (1+i\sqrt{3})(27-3\sqrt{57})^{1/3} - (1-i\sqrt{3})(27+3\sqrt{57})^{1/3}}{6}\right)^n \\
& + c_4 \left(\frac{6 - (1-i\sqrt{3})(27-3\sqrt{57})^{1/3} - (1+i\sqrt{3})(27+3\sqrt{57})^{1/3}}{6}\right)^n \\
& + c_5 \left(\frac{3 + (27-3\sqrt{57})^{1/3} + (27+3\sqrt{57})^{1/3}}{3}\right)^n,
\end{aligned}$$

Enumeration

There are exactly $\frac{1}{2} \cdot 2^n$ threshold graphs on n vertices.

$$a_n \sim \frac{1}{4}(1+\sqrt{2})^n$$

and

$$b_n \sim c_5 \left(\frac{3 + (27 - 3\sqrt{57})^{1/3} + (27 + 3\sqrt{57})^{1/3}}{3} \right)^n,$$

so for large n,

$$a_n \geq \frac{1}{4} \cdot 2.4^n \quad \text{and} \quad b_n \geq 0.096 \cdot 2.7^n.$$

Weakly threshold sequences and graphs

- Equality or a difference of 1 in each of the first $m(d)$ Erdős–Gallai inequalities.

  $k(k-1) + \sum_{i>k} \min\{k, d_i\} - \sum_{i \leq k} d_i \leq 1$

  Call these **weakly threshold sequences**; call the associated graphs **weakly threshold graphs**.

- **Iterative construction** $\boxed{\checkmark}$

- **How many weakly threshold sequences/graphs on n vertices?** $\boxed{\checkmark}$

- **Forbidden subgraph characterization** $\boxed{\checkmark}$

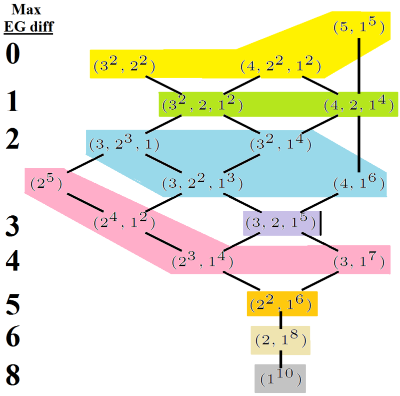
- **Unique realizations of degree sequences?** **NO**

- **Majorization result** $\boxed{\checkmark}$

- **...?...**

Further questions

Many of the results for weakly threshold graphs appear to generalize to graphs with Erdős–Gallai differences bounded by b. Do they all?



Thank you!

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