Adjacency relationships forced by graph degree sequences

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Department of Mathematics
Brigham Young University

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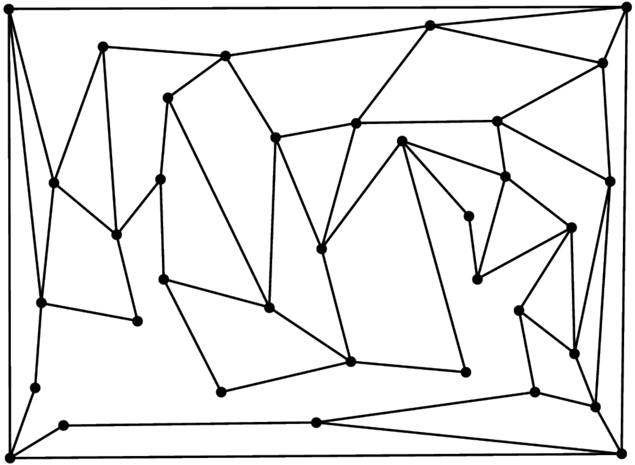
Graphs, degrees, and realizations

G:



Graphs, degrees, and realizations

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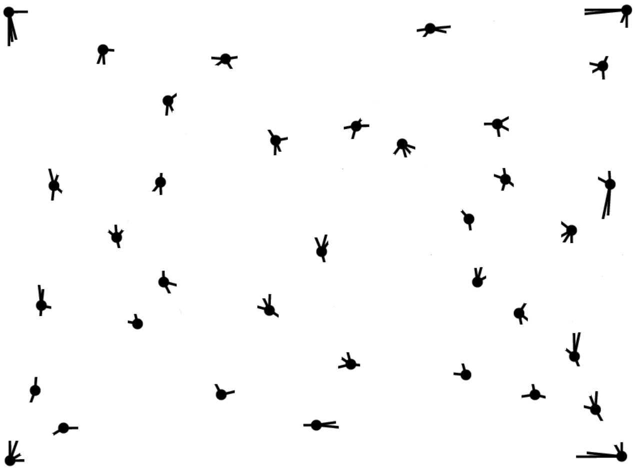

$$d(G) = (4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,\\ 3,3,3,3,3,3,3,3,2,2,2,2,2,2)$$

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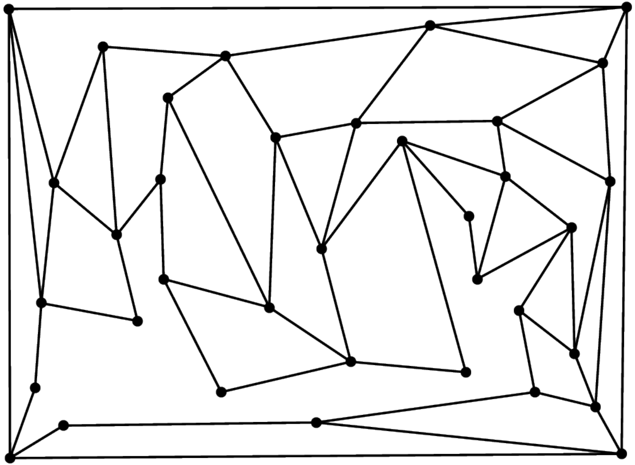

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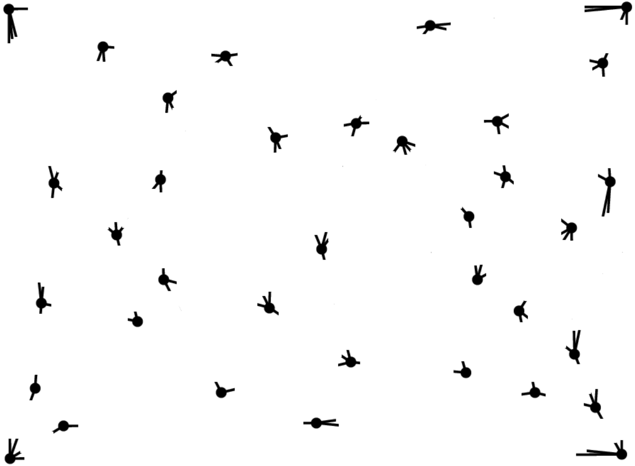

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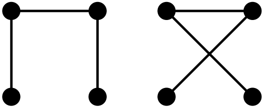
The question

$d = (2, 2, 1, 1)$



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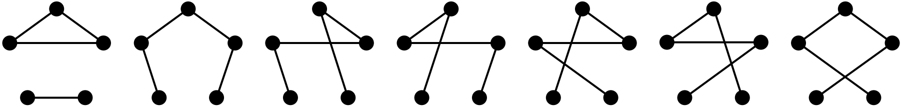
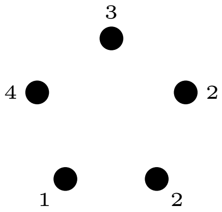


Which edges and non-edges are forced by the degree sequence?

Threshold sequences and threshold graphs

Threshold sequence: a degree sequence having exactly one realization.



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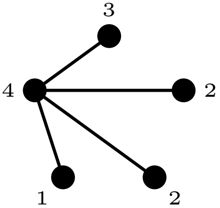

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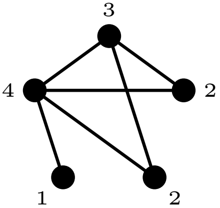

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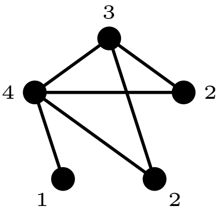
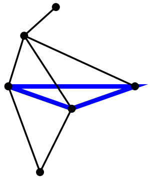


All edges and non-edges are forced by the degree sequence.

An answer

A clique is **demanding** if every vertex outside the clique has as many neighbors as possible in the clique.



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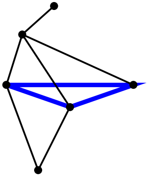
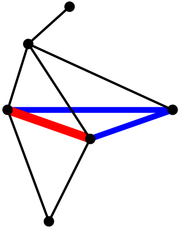
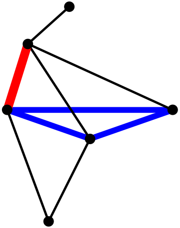
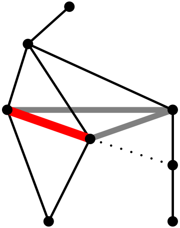


Theorem

Forced edges correspond exactly to three cases:







Threshold sequences and threshold graphs

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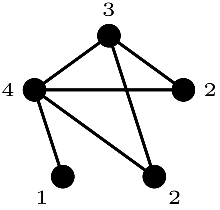

All edges and non-edges are forced by the degree sequence.

Measures of “thresholdness”

Erdős–Gallai inequalities for degree sequences $(d_1, \ldots, d_n)$**:**

$$\underbrace{\sum_{i \le k} d_i}_{\text{LHS}(k)} \le \underbrace{k(k-1) + \sum_{i > k} \min\{k, d_i\}}_{\text{RHS(k)}}$$

for all $k \le m = \max\{i : d_i \ge i - 1\}$.

Threshold graphs: $\text{RHS}(k) - \text{LHS}(k) = 0$ for all $k \in \{1, \ldots, m\}$.

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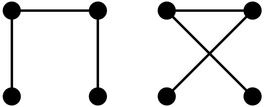
Theorem

A sequence d has ***no*** *forcible edges if and only if* $\text{RHS}(1) - \text{LHS}(1) \ge 1$ *and* $\text{RHS}(k) - \text{LHS}(k) \ge 2$ *for all* $k \in \{2, \ldots, m\}$.

Measures of “thresholdness”

$d = (2, 2, 1, 1)$



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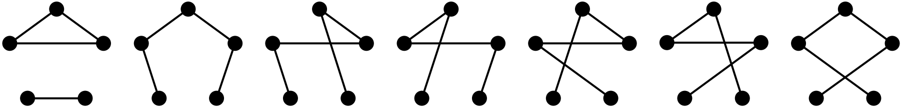

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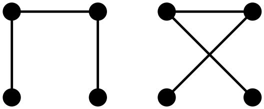
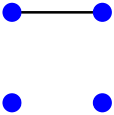
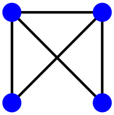


I_d:



U_d:



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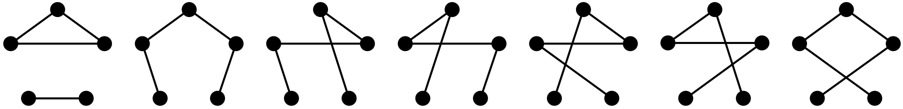
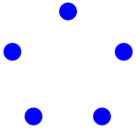


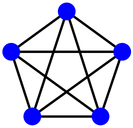
Intersection envelope graph

I_d:



Union envelope graph

U_d:

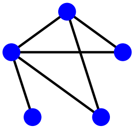


Measures of "thresholdness"

Observation

If G is threshold iff $I_{d(G)} = G = U_{d(G)}$.

I_d:



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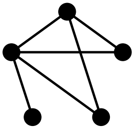
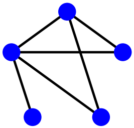


U_d:



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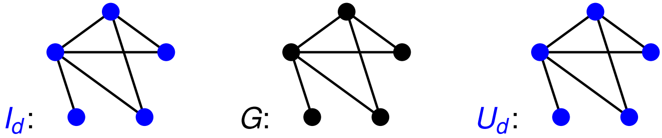
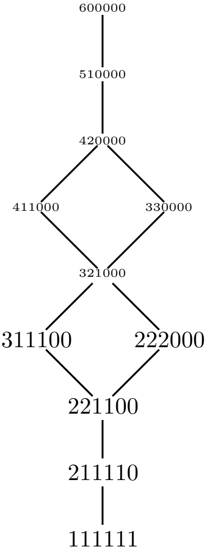


Theorem

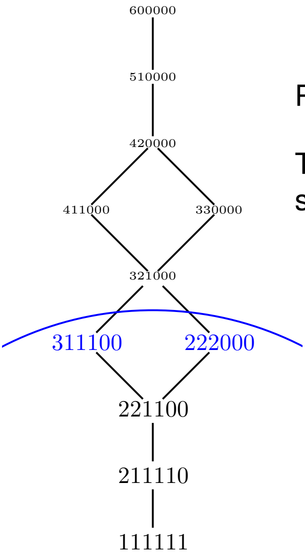
For any degree sequence d, both I_d *and* U_d *are threshold graphs.*

Forced relationships and the dominance order



Partitions of $2m$ under the dominance order

Forced relationships and the dominance order



Partitions of $2m$ under the dominance order

Threshold sequences: maximal degree sequences

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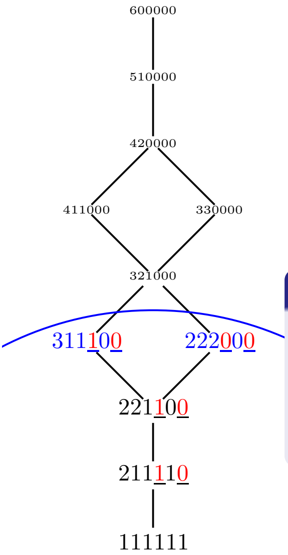

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Threshold sequences: maximal degree sequences

Theorem

If vertices i and j have a forcible adjacency relationship in realizations of d, then i and j have the same adjacency relationship for all degree sequences that majorize d.

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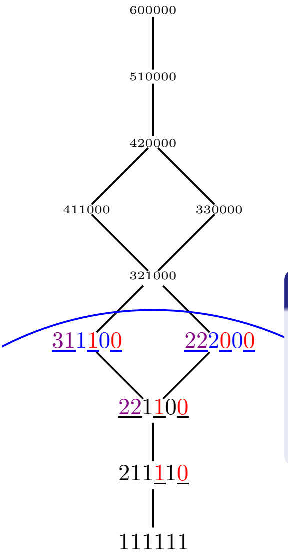

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Thank you!

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