Degree Sequences and Forced Adjacency Relationships

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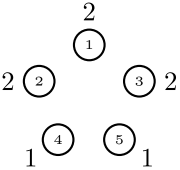
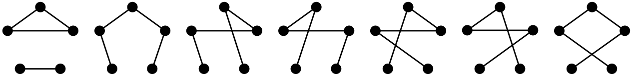
Realizations and Properties

$(2, 2, 2, 1, 1)$





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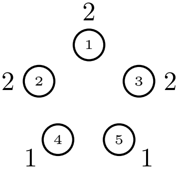
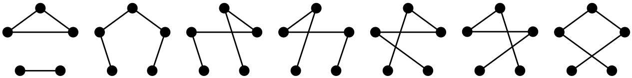
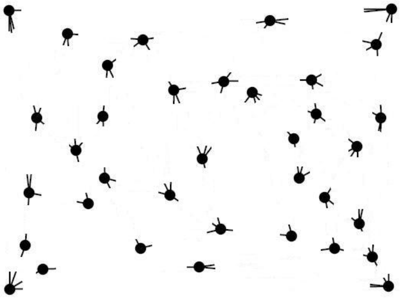




Given a graph property $\mathcal{P}$, a degree sequence *d* is

- **potentially $\mathcal{P}$-graphic** if at least one realization of *d* has property $\mathcal{P}$.
- **forcibly $\mathcal{P}$-graphic** if **every** realization of *d* has property $\mathcal{P}$.

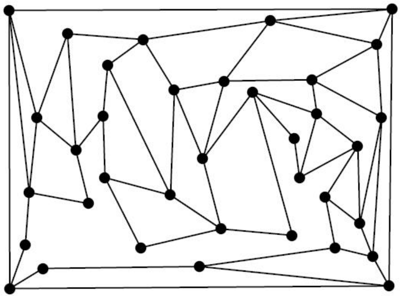
Forcible adjacency relationships



$$d(G) = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, \\ 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2)$$

Forcible adjacency relationships

$\mathcal{P}_{ij}$: *ij* is an edge (non-edge)



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Are there any forcible edges/non-edges?

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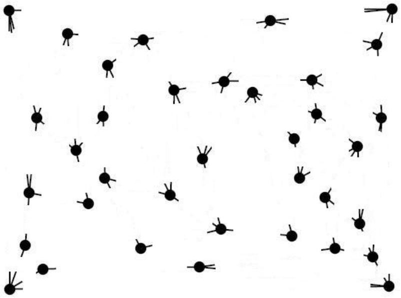

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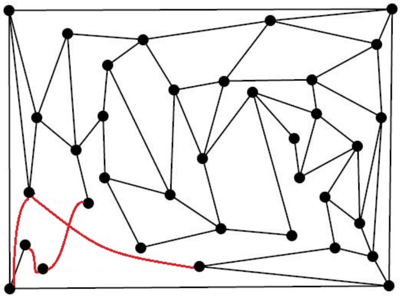

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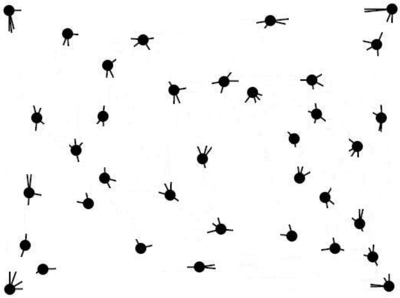

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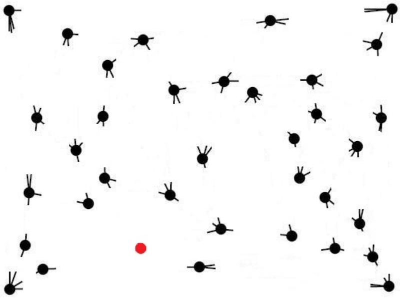

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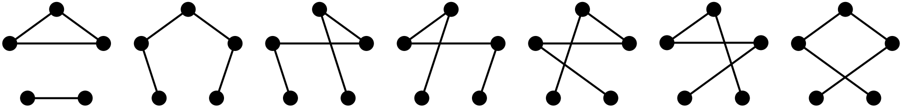
Are there any forcible edges/non-edges?

Forcible adjacency relationships: Envelope graphs

$d = (2, 2, 2, 1, 1)$



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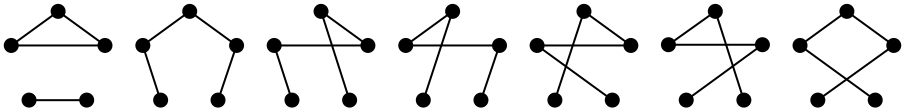
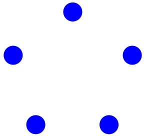


Intersection envelope graph I_d

$$E(I_d) = \bigcap_{d(G)=d} E(G)$$



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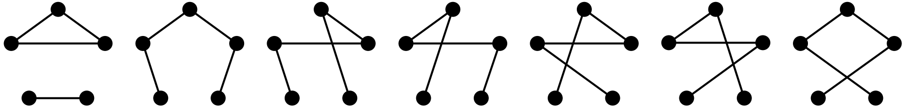

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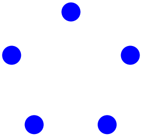
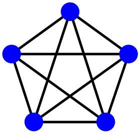
Union envelope graph U_d

$$E(U_d) = \bigcup_{d(G)=d} E(G)$$

I_d:



U_d:



Forcible adjacency relationships: Envelope graphs

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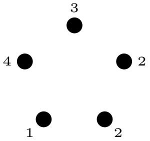

Threshold sequence **[Chvátal–Hammer, 1973]**: a degree sequence having exactly one (labeled) realization.

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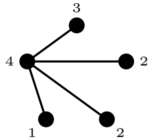

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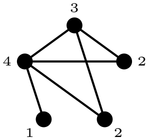

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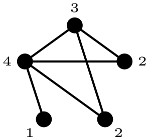
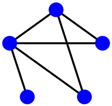
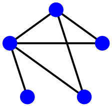


I_d: 

U_d: 

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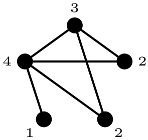
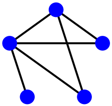
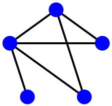


I_d: 

U_d: 

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All edges and non-edges are forced by the degree sequence.

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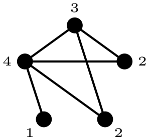
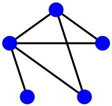
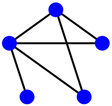


I_d: 

U_d: 

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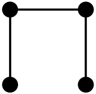

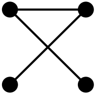

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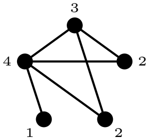
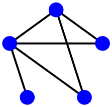
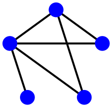


I_d:  U_d: 

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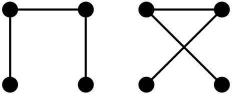
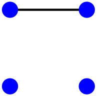
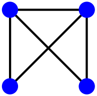


I_d:  U_d: 

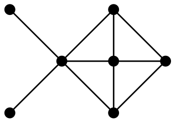
Questions

How can we recognize forcible adjacency relationships...

...from a degree sequence?

$$d = (5, 4, 3, 3, 3, 1, 1)$$

...from a graph?

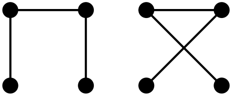


A beginning

For graphic d and $1 \leq i < j \leq n$, define

$$d^+(i,j) = (d_1, \ldots, d_{i-1}, d_i + 1, d_{i+1}, \ldots, d_{j-1}, d_j + 1, d_{j+1}, \ldots, d_n) \quad \text{and}$$
$$d^-(i,j) = (d_1, \ldots, d_{i-1}, d_i - 1, d_{i+1}, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_n).$$

$d = (2,2,1,1)$ $\quad d^+(1,3) = (3,2,2,1)$ $\quad d^+(1,2) = (3,3,1,1)$



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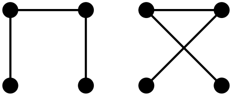

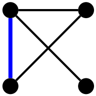

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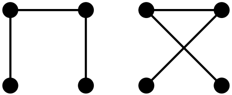

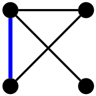

Lemma

The pair i,j is a forcible $\left\{\begin{matrix} \text{edge} \\ \text{non-edge} \end{matrix}\right\}$ for d iff $\left\{\begin{matrix} d^+(i,j) \\ d^-(i,j) \end{matrix}\right\}$ is not graphic.

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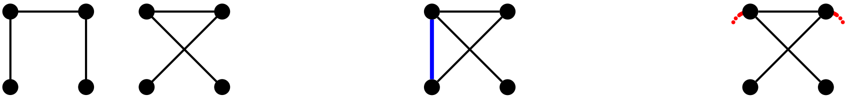

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Erdős–Gallai inequalities

A list $(d_1, \ldots, d_n)$ of nonnegative integers in descending order with even sum is a degree sequence if and only if

$$\underbrace{\sum_{i \leq k} d_i}_{\text{LHS}(k)} \leq \underbrace{k(k-1) + \sum_{i > k} \min\{k, d_i\}}_{\text{RHS(k)}}$$

for all $k \leq m = \max\{i : d_i \geq i - 1\}$.

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Theorem (Hammer–Ibaraki–Simeone, 1978)

d is a threshold sequence if and only if $\text{LHS}(k) = \text{RHS}(k)$ *for all* $k \in \{1, \dots, m\}$.

Erdős–Gallai differences

A list $(d_1, \ldots, d_n)$ of nonnegative integers in descending order with even sum is a degree sequence if and only if

$$\underbrace{\sum_{i \le k} d_i}_{\text{LHS}(k)} \le \underbrace{k(k-1) + \sum_{i>k} \min\{k, d_i\}}_{\text{RHS(k)}} \qquad \Delta(k) = \text{RHS}(k) - \text{LHS}(k)$$

for all $k \le m = \max\{i : d_i \ge i - 1\}$.

Theorem

Given $1 \le i < j \le n$*,*

$\{i, j\}$ *is a* ***forced edge*** *iff* $\exists k \in \{1, \ldots, n\}$ *such that either*

$\Delta_k(d) = 0$*,* $i \le k < j$*, and* $k \le d_j$*;* *or* $\Delta_k(d) \le 1$ *and* $j \le k$*.*

$\{i, j\}$ *is a* ***forced non-edge*** *iff* $\exists k \in \{1, \ldots, n\}$ *such that either*

$\Delta_k(d) = 0$*,* $k < i$*, and* $d_j < k \le d_i$*;* *or* $\Delta_k(d) \le 1$ *and* $d_i < k < i$*.*

(7, 6, <u>3</u>, <u>3</u>, <u>3</u>, <u>3</u>, 1, 1, 1)

(4, 4, 3, 3, 3, 1)

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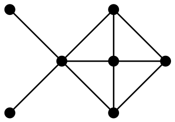
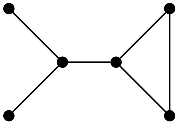


A switching result?



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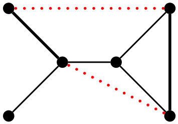
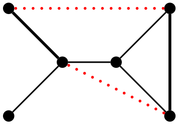
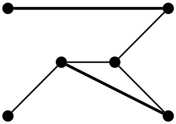


A switching result?





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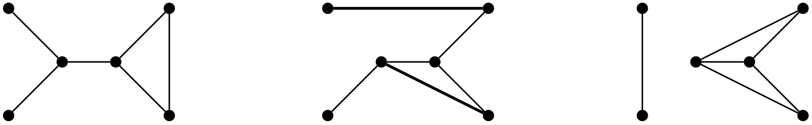

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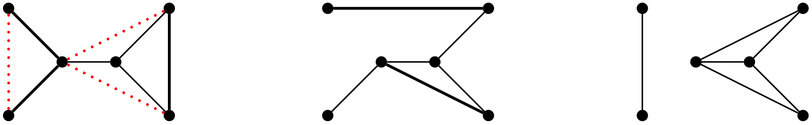

Proposition

The pair $\{i, j\}$ in G is a forcible edge or non-edge for $d(G)$ if and only if $\{i, j\}$ belongs to no alternating circuit in G.

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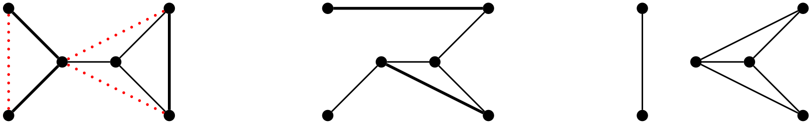

Proposition

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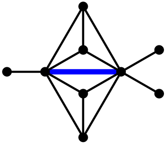
Lots to check...

A structural characterization

A clique is **demanding** if every vertex outside the clique has as many neighbors as possible in the clique.



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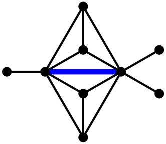

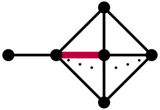

A clique is **weakly demanding** if changing one neighbor of a single vertex outside the clique makes the clique demanding.

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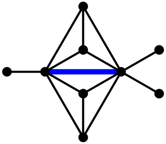
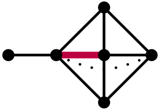
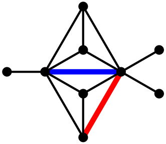




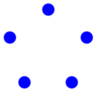
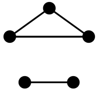
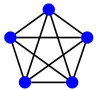
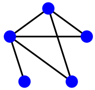
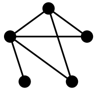
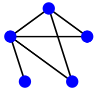
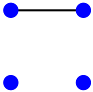
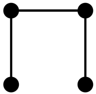
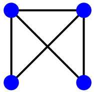
A clique is **weakly demanding** if changing one neighbor of a single vertex outside the clique makes the clique demanding.

Theorem

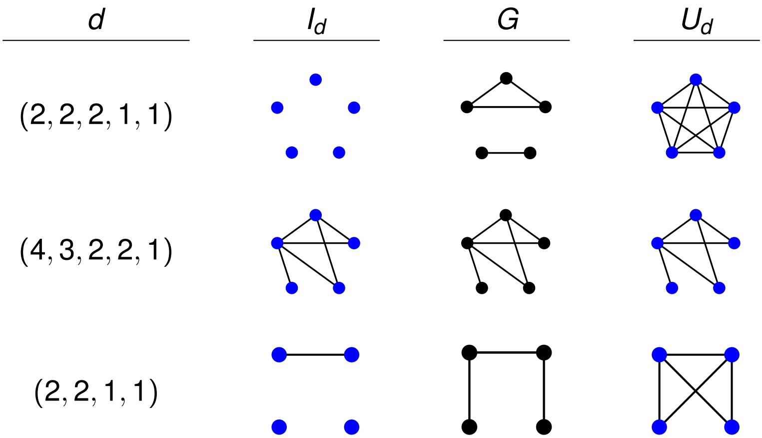
A realization edge is forced for d iff it lies in a demanding or weakly demanding clique or it joins a demanding clique vertex to an external vertex that dominates the clique.



Overall structure of forced relationships

d	I_d	G	U_d
$(2, 2, 2, 1, 1)$			
$(4, 3, 2, 2, 1)$			
$(2, 2, 1, 1)$			

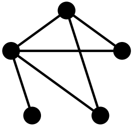
Overall structure of forced relationships



Theorem

For any degree sequence d, both I_d and U_d are threshold graphs.

Threshold graphs and canonical decomposition



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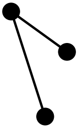

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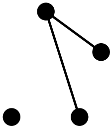

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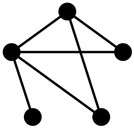

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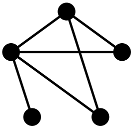

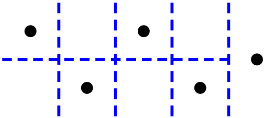

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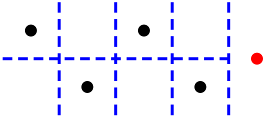

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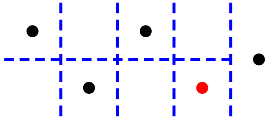

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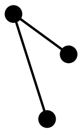

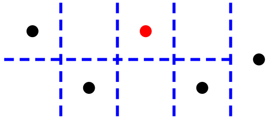

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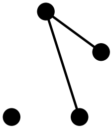

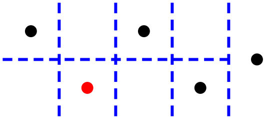

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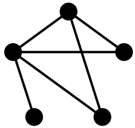

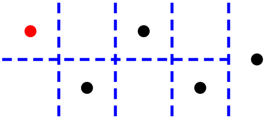

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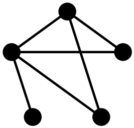

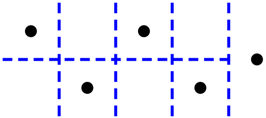

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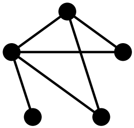

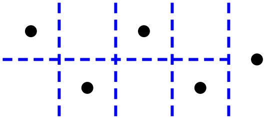

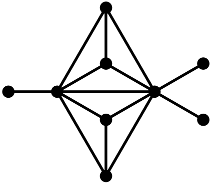

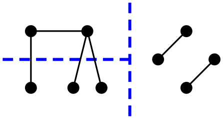

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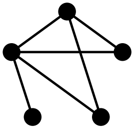
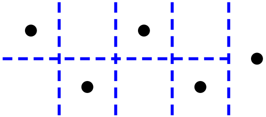
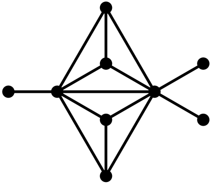
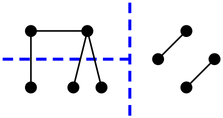


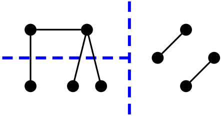


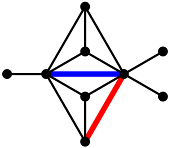




Canonical decomposition **[Tyshkevich et al., 1980's, 2000]**: Indecomposable split components hooked to each other and an indecomposable "core" following the rightwards dominating/isolated rule; every graph has a unique decomposition, up to isomorphism of canonical components.

Canonical decomposition and forced adjacency relationships





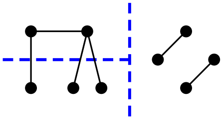
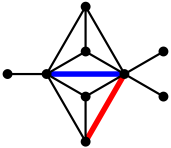
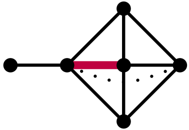
Theorem

For $k \leq m$, the following are equivalent:

- $\text{LHS}(k) = \text{RHS}(k)$;
- *Vertices* $1, \ldots, k$ *comprise a demanding clique;*
- *Vertices* $1, \ldots, k$ *comprise an initial segment of upper cells in a canonical decomposition.*

Hence all adjacency relationships between vertices in distinct canonical components are forced.

Canonical decomposition and forced adjacency relationships

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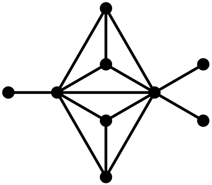
Overall structure of forced relationships

Theorem

For any degree sequence d, both I_d and U_d are threshold graphs.



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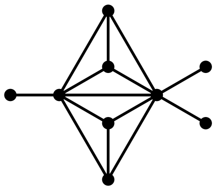

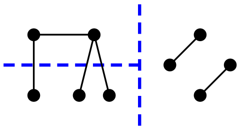

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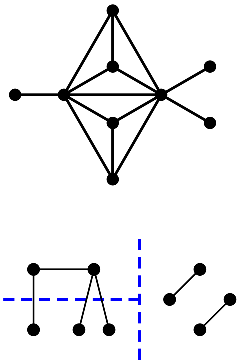

Composing the appropriate envelopes of the individual canonical components, we obtain I_d and U_d.

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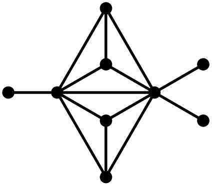

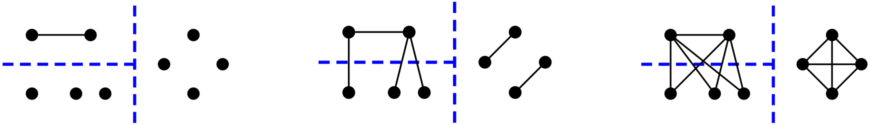

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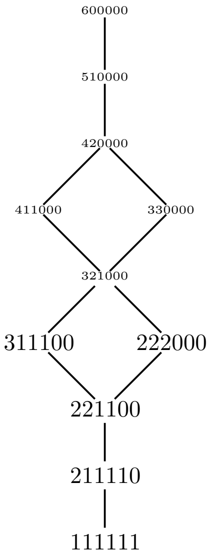
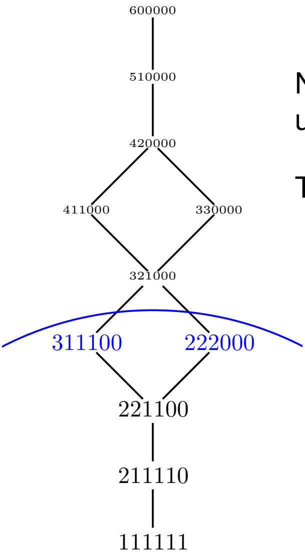
Forced relationships and the dominance order



Nonnegative partitions of $2m$ of a fixed length, under the dominance order

Forced relationships and the dominance order



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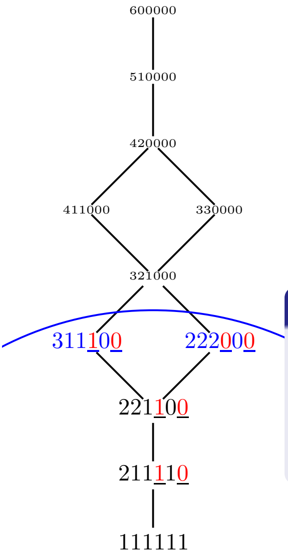
Threshold sequences: maximal graphic elements

Forced relationships and the dominance order



Nonnegative partitions of $2m$ of a fixed length, under the dominance order

Threshold sequences: maximal graphic elements

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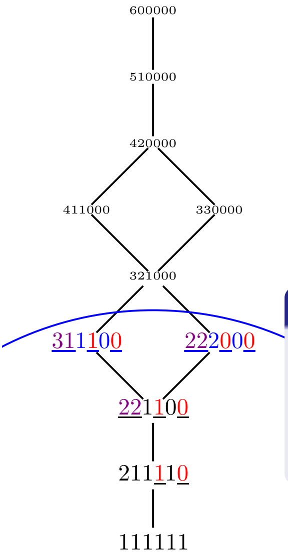
If vertices i and j have a forcible adjacency relationship in realizations of d, then i and j have the same adjacency relationship for all degree sequences that majorize d.

Forced relationships and the dominance order



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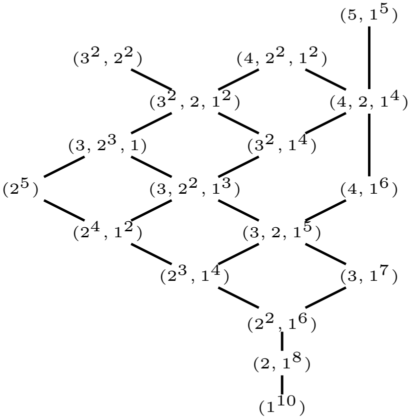
Majorization-closed classes

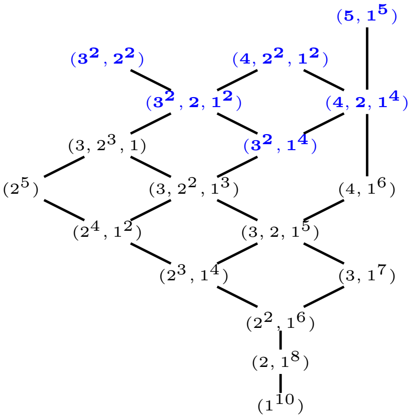


Corollary

Degree sequences for the following classes are "upwards closed" in the poset:

- *[Merris, 2003] Split graphs*
- *Canonically decomposable graphs*

Majorization-closed classes



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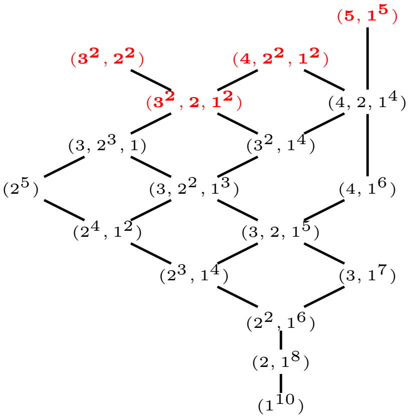

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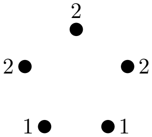
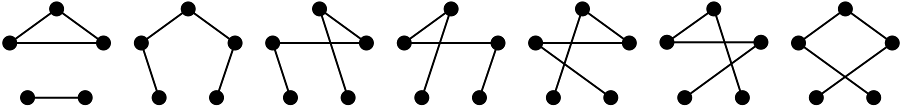
Questions

$d = (2, 2, 2, 1, 1)$





- For which degree sequences is there a simple way to compute the **probability** that two vertices are adjacent?
- What about forcing induced **subgraphs** in **unlabeled** realizations? (Can you find a forcibly P_7-inducing-graphic sequence?)