Alternating 4-cycles in graphs

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Joint work with Douglas B. West
Alternating 4-cycles...

Alternating 4-cycle ($A_4$)
Alternating 4-cycles...

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Alternating 4-cycles...

Alternating 4-cycle ($A_4$)

\begin{figure}
\centering
\begin{tikzpicture}
  \node (A) at (0,0) [circle,fill,inner sep=2pt] {};
  \node (B) at (1,0) [circle,fill,inner sep=2pt] {};
  \node (C) at (1,1) [circle,fill,inner sep=2pt] {};
  \node (D) at (0,1) [circle,fill,inner sep=2pt] {};
  \draw (A) -- (B);
  \draw (B) -- (C);
  \draw (C) -- (D);
  \draw (D) -- (A);
\end{tikzpicture}
\end{figure}
Alternating 4-cycles...

Alternating 4-cycle ($A_4$)
Alternating 4-cycles...

Alternating 4-cycle \((A_4)\)
Alternating 4-cycles...

Alternating 4-cycle \((A_4)\)

\[
\begin{align*}
\text{2K}_2 & \quad \text{P}_4 & \quad \text{C}_4 \\
\end{align*}
\]
...and degree sequences

2-switches

\[ \text{deg}(G) = \text{deg}(H) \text{ iff 2-switches transform } G \text{ into } H. \]
...and graph classes

- **Threshold graphs** (Chvátal–Hammer, 1973)
  No $A_4$’s present.

- **Matrogenic graphs** (Földes–Hammer, 1976)
  Vertex sets of $A_4$’s are circuits of a matroid on $V$.

- **Matroidal graphs** (Peled, 1977)
  Edge sets of $A_4$’s are circuits of a matroid on $E$. 
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Questions

What structural properties of a graph can we tie to the existence and location of alternating 4-cycles?

How do these affect the degree sequence?
The $A_4$-structure of a graph

Hypergraph $H$

\[ V(H) = V(G), \quad E(H) = \{ A \subseteq V(G) : G[A] \cong 2K_2 \text{ or } C_4 \text{ or } P_4 \} \]
Characterizations in terms of the $A_4$-structure

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Characterizations in terms of the $A_4$-structure

- **Threshold graphs** (Chvátal–Hammer, 1973)
  No $A_4$'s present.
  The $A_4$-structure has no edges.

- **Matrogenic graphs** (Földes–Hammer, 1976)
  Vertex sets of $A_4$'s are circuits of a matroid on $V$.
  No 5 vertices induce exactly 2 or 3 edges in the $A_4$-structure.

- **Matroidal graphs** (Peled, 1977)
  Edge sets of $A_4$'s are circuits of a matroid on $E$.
  No 5 vertices induce more than 1 edge in the $A_4$-structure.
Graphs with a common $A_4$-Structure

What properties does the $A_4$-structure determine?
Theorem

Let G and H be triangle-free graphs with the same vertex set V. If G and H have the same $A_4$-structure and $W \subseteq V$, then W is the vertex set of a nontrivial matching in one of these graphs if and only if it is in the other.

Corollary

Two triangle-free graphs with the same $A_4$-structure have maximum matchings of the same size.
Matchings

A nontrivial matching is a set of at least two pairwise non-intersecting edges.

**Theorem**

Let $G$ and $H$ be triangle-free graphs with the same vertex set $V$. If $G$ and $H$ have the same $A_4$-structure and $W \subseteq V$, then $W$ is the vertex set of a nontrivial matching in one of these graphs if and only if it is in the other.

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A *nontrivial matching* is a set of at least two pairwise non-intersecting edges.

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**Corollary**

Two triangle-free graphs with the same $A_4$-structure have maximum matchings of the same size.
Perfect graphs

The **chromatic number** of a graph is the minimum number of colors needed to properly color the graph.

The **clique number** is the size of largest set of pairwise adjacent vertices.

A graph is **perfect** if in every induced subgraph the chromatic number equals the clique number.

---

**Theorem**

*If G and H have the same $A_4$-structure, then G is perfect iff H is perfect.*
$P_4$ and modules

$2K_2$, $P_4$, $C_4$
A *module* is a vertex subset $S$ such that each vertex outside $S$ is joined to $S$ by either all possible edges or no edges.
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**Theorem**

- An induced $P_4$ intersects a module in exactly 0, 1, or 4 vertices.
- (Seinsche, 1974) *In a graph $G$ every induced subgraph on at least 3 vertices contains a nontrivial module iff $G$ is $P_4$-free.*
A module $S$ is a vertex subset such that no alternating path of length 2 begins and ends in $S$ and has its midpoint outside $S$. 

Forbidden:
Strict modules

Define a **strict module** to be a vertex subset $S$ such that no (possibly closed) alternating path of length 2 or 3 begins and ends in $S$ and has its midpoints outside $S$.

Forbidden:

This is equivalent to not having alternating paths of *any* length begin and end in $S$. 
A_4 and strict modules

**Lemma**

An A_4 intersects a strict module in exactly 0 or 4 vertices.

**Theorem**

In a graph G every induced subgraph on at least 2 vertices has a nontrivial strict module if and only if G is A_4-free, i.e., threshold.
Strict modules and graph structure

Forbidden:
Lemma

The vertices which dominate a strict module form a clique, and those which are nonadjacent to the strict module form an independent set.
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Iterate to get a “strict modular decomposition”? 
Decomposition
Canonical decomposition

**Theorem (Tyshkevich–Chernyak, 1978; Tyshkevich, 2000)**

Every graph $F$ can be represented as a composition

$$F = (G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ F_0$$

of indecomposable components. Here the $(G_i, A_i, B_i)$ are indecomposable splitted graphs and $F_0$ is an indecomposable graph. This decomposition is unique up to isomorphism of components.
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Where did this come from?
Previous motivation for canonical decomposition

Just seemed to show up...

- **Matrogenic graphs** (Földes–Hammer, 1976; Tyshkevich, 1984)
- **Unigraphs** (Tyshkevich–Chernyak, 1978–1979)
- **Box-threshold graphs** (Tyshkevich–Chernyak, 1985)
- **Pseudo-split graphs** (Blázsik et al., 1993)

In each case, indecomposable components restricted to certain classes.

Structural properties lead to degree sequence characterizations.
Lemma

An $A_4$ intersects a strict module in exactly 0 or 4 vertices.

Forbidden:

Theorem

In a graph $G$ every induced subgraph on at least 2 vertices has a nontrivial strict module if and only if $G$ is $A_4$-free, i.e., threshold.
A graph is indecomposable in the canonical decomposition if and only if its \(A_4\)-structure is connected.

Hence the components of the \(A_4\)-structure and of the canonical decomposition partition the vertex set in the same way.
Motivation for canonical decomposition

- Graph classes (matrogenic, unigraphs, etc.)
- Strict modular decomposition
- Components of the $A_4$-structure
A graph is indecomposable in the canonical decomposition if and only if its $A_4$-structure is connected.

Hence the components of the $A_4$-structure and of the canonical decomposition partition the vertex set in the same way.
Beginnings

Lemma

The graphs $2K_2$, $C_4$, and $P_4$ are all indecomposable. Therefore, connected $A_4$-structure $\Rightarrow$ indecomposable.

Forbidden:

Lemma

In an indecomposable graph $G$ with more than 1 vertex, every vertex belongs to an alternating 4-cycle.
Disjoint $A_4$s

Lemma

If $A$ and $B$ are disjoint alternating 4-cycles in $G$ such that no third alternating cycle in $G$ intersects each, then either $A$ induces $P_4$, with its interior vertices dominating $B$ and the endpoints isolated from $B$ (denote this by $A \rightarrow B$), or vice versa.
Disjoint $A_4$s

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More on disjoint \( A_4 \)s

**Corollary**

Any two vertices which both belong to induced \( 2K_2 \)'s or \( C_4 \)'s have distance at most 3 in the \( A_4 \)-structure.

**Lemma**

The \( \rightarrow \) relation is consistent among pairs of \( A_4 \)s from different components of the \( A_4 \)-structure.
Putting it all together

Lemma

The tournament on the $A_4$-components of a graph is acyclic.
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The tournament on the $A_4$-components of a graph is acyclic.

Having a source implies the graph is decomposable.

$\therefore \text{not } A_4\text{-connected } \implies \text{decomposable.}$
Putting it all together

Lemma

The → tournament on the $A_4$-components of a graph is acyclic.

Having a source implies the graph is decomposable.

$\therefore A_4$-connected $\iff$ indecomposable.
Questions

What structural properties of a graph can we tie to the existence and location of alternating 4-cycles?

How do these affect the degree sequence?
Degree sequence connections

**Theorem (Erdős–Gallai, 1960)**

Let $d = (d_1, \ldots, d_n)$ be a list of nonnegative integers with even sum, arranged in descending order. $d$ is the degree sequence of a simple graph if and only if for all $k$,

$$
\sum_{i \leq k} d_i \leq k(k - 1) + \sum_{i > k} \min\{k, d_i\}.
$$
Theorem (B, 2013)

Let $d$ be the degree sequence of $G$. The graph $G$ is canonically indecomposable if and only if $d_n > 0$ and no Erdős–Gallai inequality holds with equality.

Moreover, by examining the values $k$ for which the $k$th inequality is an equality, we can determine the sizes of the “cells” in the canonical decomposition.

Corollary

Knowing which Erdős–Gallai inequalities hold with equality (and the multiplicity of 0 as a term in $d$) is equivalent to knowing the vertex sets of the $A_4$-structure components.
Future applications of the $A_4$-structure

Characterizations of graph/degree sequence properties

- Graph classes (threshold, matrogenic, etc.)
- Matchings
- Perfection
- Strict modules/canonical decomposition
- Erdős–Gallai inequalities
- ?
What other properties does the $A_4$-structure determine?
Graphs with a common $A_4$-Structure

What other properties does the $A_4$-structure determine?

Which graphs have the same $A_4$-structure?
Obtaining other realizations: decomposable graphs

- 
- 
- 
- 

Alternating 4-cycles in graphs

M. D. Barrus (BYU)
Obtaining other realizations: decomposable graphs

\begin{itemize}
\item Graph 1
\item Graph 2
\item Graph 3
\item Graph 4
\end{itemize}
Obtaining other realizations: decomposable graphs

\[ \text{Graph 1} \quad \text{Graph 2} \]

\[ \begin{array}{c}
\text{\includegraphics[width=0.4\textwidth]{Graph1.png}} \\
\text{\includegraphics[width=0.4\textwidth]{Graph2.png}} 
\end{array} \]
Obtaining other realizations: decomposable graphs

\begin{center}
\begin{tikzpicture}
  \node at (0,0) [circle, fill, inner sep=0.5mm] (a) {};
  \node at (0.5,0) [circle, fill, inner sep=0.5mm] (b) {};
  \node at (1,0) [circle, fill, inner sep=0.5mm] (c) {};
  \node at (1.5,0) [circle, fill, inner sep=0.5mm] (d) {};
  \node at (2,0) [circle, fill, inner sep=0.5mm] (e) {};
  \node at (2.5,0) [circle, fill, inner sep=0.5mm] (f) {};
  \node at (3,0) [circle, fill, inner sep=0.5mm] (g) {};
  \node at (3.5,0) [circle, fill, inner sep=0.5mm] (h) {};
  \node at (0,1) [circle, fill, inner sep=0.5mm] (i) {};
  \node at (0.5,1) [circle, fill, inner sep=0.5mm] (j) {};
  \node at (1,1) [circle, fill, inner sep=0.5mm] (k) {};
  \node at (1.5,1) [circle, fill, inner sep=0.5mm] (l) {};
  \node at (2,1) [circle, fill, inner sep=0.5mm] (m) {};
  \node at (2.5,1) [circle, fill, inner sep=0.5mm] (n) {};
  \draw (a) -- (b) -- (c) -- (d) -- (e) -- (f) -- (g) -- (h);
  \draw (i) -- (j) -- (k) -- (l) -- (m) -- (n);
  \draw (a) -- (j);
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\end{tikzpicture}
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  \node at (3.5,1) [circle, fill, inner sep=0.5mm] (p) {};
  \node at (4,1) [circle, fill, inner sep=0.5mm] (q) {};
  \node at (4.5,1) [circle, fill, inner sep=0.5mm] (r) {};
  \node at (5,1) [circle, fill, inner sep=0.5mm] (s) {};
  \node at (5.5,1) [circle, fill, inner sep=0.5mm] (t) {};
  \node at (6,1) [circle, fill, inner sep=0.5mm] (u) {};
  \node at (6.5,1) [circle, fill, inner sep=0.5mm] (v) {};
  \node at (7,1) [circle, fill, inner sep=0.5mm] (w) {};
  \node at (7.5,1) [circle, fill, inner sep=0.5mm] (x) {};
  \node at (8,1) [circle, fill, inner sep=0.5mm] (y) {};
  \node at (8.5,1) [circle, fill, inner sep=0.5mm] (z) {};
  \draw (a) -- (b) -- (c) -- (d) -- (e) -- (f) -- (g) -- (h);
  \draw (i) -- (j) -- (k) -- (l) -- (m) -- (n) -- (o) -- (p) -- (q) -- (r) -- (s) -- (t) -- (u) -- (v) -- (w) -- (x) -- (y) -- (z);
  \draw (a) -- (j);
  \draw (i) -- (h);
\end{tikzpicture}
\end{center}
Obtaining other realizations: decomposable graphs
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The rightmost $A_4$-component may only be transposed if it has a split realization.

Which graphs have the same $A_4$-structure as a split graph?
A₄-split graphs

Theorem

A graph is A₄-split iff each canonical component is. For an indecomposable graph G with A₄-structure H, the following are equivalent:

(i) G is A₄-split.

(ii) H is balanced and satisfies the bipartite restriction property.

(iii) G is \{C₅, P₅, house, K₂ + K₃, K₂,3, P, \overline{P}, K₂ + P₄, P₄ ∨ 2K₁, K₂ + C₄, 2K₂ ∨ 2K₁\}-free.

(iv) G is split, or G or \(\overline{G}\) is a disjoint union of stars.

(v) G is A₄-separable.
Future applications of the $A_4$-structure

Characterizations of graph/degree sequence properties

- Graph classes (threshold, matrogenic, etc.)
- Matchings
- Perfection
- Strict modules/canonical decomposition
- Erdős–Gallai inequalities
- ?
Antimagic labelings of graphs

Magic square: equal sums along each row, column, and main diagonal.
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Antimagic graph labeling: edges labeled with $1, \ldots, |E(G)|$, all vertex sums distinct.

**Conjecture** (Hartsfield–Ringel, 1990): Every connected graph other than $K_2$ has an antimagic labeling.
Antimagic labelings of graphs

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**Conjecture** (Hartsfield–Ringel, 1990): Every connected graph other than $K_2$ has an antimagic labeling.
Canonical decomposition?

Theorem (Alon et al., 2004)

If $G (\not\cong K_2)$ has a vertex which is adjacent to all other vertices, then $G$ has an antimagic labeling.

Pf:
Theorem (Alon et al., 2004)

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**Pf:**
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If $G (\not\cong K_2)$ has a vertex which is adjacent to all other vertices, then $G$ has an antimagic labeling.

Pf:
Theorem (Alon et al., 2004)

If \( G \neq K_2 \) has a vertex which is adjacent to all other vertices, then \( G \) has an antimagic labeling.

\[ 5 \quad 1 \quad 4 \]

\[ 4 \quad 2 \quad 6 \]

\[ 3 \quad 5 \]

\( Pf: \)
Theorem (Alon et al., 2004)

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Pf:

\begin{align*}
1 & \quad 1 \\
2 & \quad 3 \\
4 & \quad 6 \\
5 & \quad 8 \\
7 & \quad 26 \\
9 & \\
11 & \\
12 & \\
14 & \\
\end{align*}
Theorem (B, 2010)

If connected $G \not\cong K_2$ is split or canonically decomposable, then $G$ has an antimagic labeling.

Pf. sketch:
Theorem (B, 2010)

If connected $G (\not\cong K_2)$ is split or canonically decomposable, then $G$ has an antimagic labeling.

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Pf. sketch:
Theorem (B, 2010)

*If connected* $G \not\cong K_2$ *is split or canonically decomposable, then* $G$ *has an antimagic labeling.*

Pf. sketch:
Theorem (B, 2010)

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Pf. sketch:
**Theorem (B, 2010)**

*If connected $G (\not\simeq K_2)$ is split or canonically decomposable, then $G$ has an antimagic labeling.*

**Pf. sketch:**

\[ \text{Diagram showing antimagic labeling with numbers assigned to vertices and edges.} \]
Theorem (B, 2010)

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True conjecture: how to label!
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Pf. sketch:

Possible $A_4$-structure help?
True conjecture: how to label!

False conjecture: counterexample!
Future applications of the $A_4$-structure

Characterizations of graph/degree sequence properties

- Graph classes (threshold, matrogenic, etc.)
- Matchings
- Perfection
- Strict modules/canonical decomposition
- Erdős–Gallai inequalities
- ?