Colored Saturation Parameters for Rainbow Subgraphs

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Abstract

Inspired by a 1987 result of Hanson and Toft [Edge-colored saturated graphs, *J. Graph Theory* **11** (1987), 191–196] and several recent results, we consider the following saturation problem for edge-colored graphs. An edge-coloring of a graph F is *rainbow* if every edge of F receives a different color. Let $\mathfrak{R}(F)$ denote the set of rainbow-colored copies of F. A t-edge-colored graph G is $(\mathfrak{R}(F), t)$-*saturated* if G does not contain a rainbow copy of F but for any edge $e \in E(\overline{G})$ and any color $i \in [t]$, the addition of e to G in color i creates a rainbow copy of F. Let $\mathrm{sat}_t(n, \mathfrak{R}(F))$ denote the minimum number of edges in an $(\mathfrak{R}(F), t)$-saturated graph of order n. We call this the *rainbow saturation number* of F.

In this paper, we prove several results about rainbow saturation numbers of graphs. In stark contrast with the related problem for monochromatic subgraphs, wherein the saturation is always linear in n, we prove that rainbow saturation numbers have a variety of different orders of growth. For instance, the rainbow saturation number of the complete graph K_n lies between $n \log n / \log \log n$ and $n \log n$, the rainbow saturation number of an n-vertex star is quadratic in n, and the rainbow saturation number of any tree that is not a star is at most linear.

Keywords: saturation; edge-coloring; rainbow

1   Introduction

All graphs considered in this paper are simple. For a positive integer t, we let $[t]$ denote the set $\{1, \ldots, t\}$. The degree of a vertex v will be denoted $d(v)$, and the minimum and maximum degree of a graph G will be denoted $\delta(G)$ and $\Delta(G)$, respectively. A *t-edge-coloring* of a graph

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G is a function $f : E(G) \to [t]$, and a graph equipped with such a coloring is a *t-edge-colored graph*. In this paper, we do not require edge-colorings to be proper edge-colorings.

Given a family of graphs $\mathcal{F}$, a graph G is $\mathcal{F}$-*saturated* if no $F \in \mathcal{F}$ is a subgraph of G, but for any $e \in E(\overline{G})$, some $F \in \mathcal{F}$ is a subgraph of $G + e$. The minimum number of edges in an n-vertex $\mathcal{F}$-saturated graph is the *saturation number* of $\mathcal{F}$ and is denoted $\mathrm{sat}(n, \mathcal{F})$. If $\mathcal{F} = \{F\}$, then we instead say that G is F-*saturated*, and write $\mathrm{sat}(n, F)$. Saturation numbers were introduced by Erdős, Hajnal, and Moon in [3], where they determined the saturation number of K_k and characterized the n-vertex K_k-saturated graphs of this size. Since then, saturation numbers have received considerable attention; for more results we refer the reader to the dynamic survey of Faudree, Faudree, and Schmitt [5].

In [7], Hanson and Toft extended the notion of saturation numbers to edge-colored graphs. Given a family $\mathcal{C}$ of edge-colored graphs, we say that a t-edge-colored graph G is $(\mathcal{C}, t)$-*saturated* if G contains no member of $\mathcal{C}$ as a (colored) subgraph, but for any edge $e \in E(\overline{G})$ and any color $i \in [t]$, the addition of e to G in color i creates some member of $\mathcal{C}$. Let $\mathrm{sat}_t(n, \mathcal{C})$ denote the minimum number of edges in a $(\mathcal{C}, t)$-saturated graph of order n. We call this the *t-edge-colored saturation number* of $\mathcal{C}$. Following [6], we will refer to a coloring of G with this property as a $\mathcal{C}$-*threshold coloring*.

Let $\mathcal{M}(H_1, \ldots, H_t)$ denote the set containing one copy each of $H_1, \ldots, H_t$ such that for each i in $[t]$, all edges in H_i are colored with color i. In [7], Hanson and Toft proved the following theorem for the saturation number of monochromatic complete graphs.

Theorem 1 (Hanson and Toft [7]). *Let t be a positive integer. If $k_i \geq 2$ is a positive integer for $1 \leq i \leq t$ and $k = \sum k_i$, then*

$$\mathrm{sat}_t(n, \mathcal{M}(K_{k_1}, \ldots, K_{k_t})) = \begin{cases} \binom{n}{2} & \text{if } n \leq k - 2t \\ \binom{k-2t}{2} + (k - 2t)(n - k + 2t) & \text{if } n > k - 2t. \end{cases}$$

It is important to recall that $(\mathcal{C}, t)$-saturated graphs are already edge-colored. However, for Theorem 1, the condition that the addition of any edge in any color yields a monochromatic complete graph bears a striking resemblance to the main ideas of Ramsey theory, in which the graphs do not have a specified coloring.

Given graphs G and $H_1, \ldots, H_t$, we write $G \to (H_1, \ldots, H_t)$ if every t-edge coloring of G contains a copy of H_i that is monochromatic in color i for some $i \in [t]$. Thus the Ramsey number $r(k_1, \ldots, k_t)$ is the minimum n such that $K_n \to (K_{k_1}, \ldots, K_{k_t})$. A graph G is $(H_1, \ldots, H_t)$-*Ramsey-minimal* if $G \to (H_1, \ldots, H_t)$ but $G - e \nrightarrow (H_1, \ldots, H_t)$ for all $e \in E(G)$. The set of $(H_1, \ldots, H_t)$-Ramsey-minimal graphs is denoted $\mathcal{R}_{\min}(H_1, \ldots, H_t)$.

Hanson and Toft made the following conjecture.

Conjecture 1. *If $r = r(k_1, \ldots, k_t)$ is the standard Ramsey number for complete graphs, then*

$$\mathrm{sat}(n, \mathcal{R}_{\min}(K_{k_1}, \ldots, K_{k_t})) = \begin{cases} \binom{n}{2} & \text{if } n < r \\ \binom{r-2}{2} + (r-2)(n-r+2) & \text{if } n \geq r. \end{cases}$$

The upper bound follows easily from any t-edge-colored clique K of order $r-1$ that has no copy of K_{k_i} in color i for any i. Specifically, add $n-r+1$ new vertices $\{v_1, \ldots, v_{n-r+1}\}$ to K, adjacent to each vertex in $K-v$, and for each vertex $x \in K-v$ and each v_i, give $v_i x$ the same color as vx. As the addition of any edge to this graph creates a copy of K_r, any edge coloring must contain a monochromatic copy of K_i in color i for some i.

A graph that is $\mathcal{R}_{\min}(H_1, \ldots, H_t)$-saturated, which is not edge-colored, has an edge coloring that does not contain a copy of H_i for any $i \in [t]$. However, the addition of any edge e to G yields a subgraph G' such that $G' \to (H_1, \ldots, H_t)$, so that every t-edge-coloring of $G+e$ must contain a monochromatic copy of H_i in color i for some $i \in [t]$.

In [2], Chen et al. verified the Hanson–Toft conjecture for $\mathrm{sat}(n, \mathcal{R}_{\min}(K_3, K_3))$, the first nontrivial case. They also proved an upper bound on $\mathrm{sat}(n, \mathcal{R}_{\min}(K_t, T_m))$ where T_m is a tree of order m and determined $\mathrm{sat}(n, \mathcal{R}_{\min}(K_3, P_3))$. More recently [6], Ferrara, Kim and Yeager determined $\mathrm{sat}(n, \mathcal{R}_{\min}(m_1 K_2, \ldots, m_k K_2))$ for $m_1, \ldots, m_k \geq 1$ and $n > 3(m_1 + \ldots + m_k - k)$, and characterized the saturated graphs of minimum size.

In this paper we consider saturation numbers of edge-colored graphs that are as far from being monochromatic as possible. An edge-coloring of a graph F is *rainbow* if every edge of F receives a different color. Let $\mathfrak{R}(F)$ denote the set of rainbow-colored copies of F; as a technical detail, note that it is not necessary to specify the set of colors that may be used to edge-color F in $\mathfrak{R}(F)$. In this paper we study $\mathrm{sat}_t(n, \mathfrak{R}(F))$, the t-edge-colored saturation number for rainbow copies of F. Informally we refer to this as the *rainbow saturation number* of F.

Observe that for $F \neq K_2$, a monochromatic complete graph does not contain an element of $\mathfrak{R}(F)$ and therefore is vacuously $(\mathfrak{R}(F), t)$-saturated. Also, the empty graph is $(\mathfrak{R}(K_2), t)$-saturated. Therefore $\mathrm{sat}_t(n, \mathfrak{R}(F))$ is defined whenever F is nonempty.

Note that every noncomplete $\mathcal{R}_{\min}(H_1, \ldots, H_k)$-saturated graph necessarily has an $\mathcal{M}(H_1, \ldots, H_t)$-threshold coloring. Consequently,

$$\mathrm{sat}(n, \mathcal{M}(H_1, \ldots, H_t)) \leq \mathrm{sat}(n, \mathcal{R}_{\min}(H_1, \ldots, H_t)).$$

In [8], Kászonyi and Tuza proved that for any nonempty family of graphs $\mathcal{F}$, $\mathrm{sat}(n, \mathcal{F}) = O(n)$, which therefore implies that

$$\mathrm{sat}(n, \mathcal{M}(H_1, \ldots, H_t)) = O(n).$$

Our results show that this is unequivocally not the case for rainbow saturation numbers. In Section 2 we study the rainbow saturation number of complete graphs, proving that their order of growth lies between $n \log n / \log \log n$ and $n \log n$. In Section 3, we consider the rainbow saturation number of some additional graphs, including stars, whose rainbow saturation numbers are quadratic in n, and paths, whose saturation numbers are at least linear. In Section 4 we establish linear upper bounds on the rainbow saturation number of various families of graphs. These include cycles with length at least 4 and trees that are not stars. Section 5 contains several open questions and conjectures.

2 Complete Graphs

In this section we study the rainbow saturation number of complete graphs. The following theorem is the main result of this section.

Theorem 2. *Let k be a positive integer that is at least 3, and let t be an integer that is at least $\binom{k}{2}$. For all n sufficiently large, there exist constants c_1 and c_2 such that*

$$c_1 \frac{n \log n}{\log \log n} \leq \mathrm{sat}_t(n, \mathfrak{R}(K_k)) \leq c_2 n \log n.$$

Theorem 2 follows immediately from Theorems 3 and 4 below, each of which regards a broader class of graphs than cliques alone. Theorem 3, which applies to all graphs in which every edge lies on a triangle, supplies the lower bound. The upper bound follows from Theorem 4, which applies to all connected graphs that have no independent vertex cut.

Theorem 3. *Let F be a graph with the property that every edge in F lies on a triangle. For every $t \geq |E(F)|$ there exists a constant $c = c(F)$ such that*

$$\mathrm{sat}_t(n, \mathfrak{R}(F)) \geq cn \frac{\log n}{\log \log n}.$$

Proof. Suppose that G is an $(\mathfrak{R}(F), t)$-saturated graph of order n. We wish to show that G has $\Omega\left(n \frac{\log n}{\log \log n}\right)$ edges. This is a nearly immediate consequence of the following claim.

Claim 1. Let G be an edge-colored graph such that for every pair of nonadjacent vertices x and y in $V(G)$, there are two internally disjoint rainbow paths of length 2 joining x and y. For $d > t \geq 3$, G contains at most $t^{d-1}d^d$ vertices of degree at most d.

Proof. Let X be the set of vertices in G with degree at most d and assume that $|X| > t^{d-1}d^d$. We demonstrate the existence of a pair of nonadjacent vertices x and y in X that are connected by $d-1$ internally disjoint monochromatic paths of length 2. This contradicts the assumption that x and y are also connected by a pair of internally disjoint rainbow paths.

Observe that G is connected, $\Delta(G[X]) \leq d$, and $|X| \geq d+2$; hence $G[X]$ does not contain K_{d+1}. Therefore Brooks' Theorem [1] implies that $G[X]$ is d-colorable and there is an independent set $S_0 \subseteq X$ such that $|S_0| > t^{d-1}d^{d-1}$. For each vertex $v \in S_0$ with degree less than d, add $d - d(v)$ edges joining v to nonneighbors in $V(G) - S_0$. Let G' be the graph obtained by adding these edges; observe that nonadjacent vertices in G' are joined by two internally-disjoint rainbow paths of length 2.

We iteratively construct two families of nested sets $S_0 \supseteq S_1 \supseteq \ldots \supseteq S_{d-1}$ and $L_0 \subseteq L_1 \subseteq \ldots \subseteq L_{d-1}$ so that S_i and L_i satisfy the following four properties for all $i \in \{0, \ldots, d-1\}$:

(1) S_i is an independent set with $|S_i| > t^{d-1-i}d^{d-1-i}$,

(2) $|L_i| = i$,

(3) every vertex in L_i is adjacent to every vertex in S_i, and

(4) if $x \in L_i$, then every edge joining x to S_i has the same color.

Setting $L_0 = \emptyset$, it is clear that S_0 and L_0 satisfy properties (1) through (4) for $i = 0$.

Now assume that S_i and L_i satisfy properties (1) through (4) for some $i \in \{0, \ldots, d-2\}$. Pick $v_i \in S_i$ and let $N_i = N(v_i) - L_i$. For every $s \neq v_i$ in S_i, there are two internally disjoint rainbow s, v_i-paths of length 2 in G'. By property (4), every rainbow path of length 2 joining v_i and s must contain a vertex in N_i. Therefore s has at least two neighbors in N_i, and there are at least $2(|S_i| - 1)$ edges joining S_i and N_i. Let ℓ_{i+1} be a vertex in N_i that is joined to S_i with the maximum number of edges all of the same color; call this color c_{i+1}. Let S_{i+1} be the set of vertices in S_i that are adjacent to ℓ_{i+1} with an edge of color c_{i+1}. By the Pigeonhole Principle,

$$|S_{i+1}| \geq \frac{2(|S_i| - 1)}{t(d - i)} \geq \frac{|S_i|}{td} > \frac{t^{d-i-1}d^{d-i-1}}{td} = t^{d-(i+1)-1}d^{d-(i+1)-1},$$

where the second inequality follows from the fact that $|S_i| \geq 1$.

Set $L_{i+1} = L_i \cup \{\ell_{i+1}\}$. Clearly L_{i+1} satisfies properties (2) and (3). By induction, for every $\ell \in L_i$ there is a single color on all of the edges joining ℓ and S_i, and consequently S_{i+1} as well. By definition, all edges joining ℓ_{i+1} and S_{i+1} have the same color, and hence L_{i+1} satisfies condition (4).

By Property (1), we know that S_{d-1} contains at least two vertices. Furthermore, given two vertices x and y in S_{d-1}, the $d-1$ paths of length 2 joining x and y through L_{d-1} are all monochromatic. Since x and y have degree d in G', it follows that there is at most one rainbow path of length 2 joining x and y in G', and consequently also in G. □

To complete the proof of Theorem 3, we observe that since every edge in F lies in a triangle, nonadjacent vertices in any $(\mathfrak{R}(F), t)$-saturated graph must be joined by at least two internally disjoint rainbow paths of length 2. Therefore, Claim 1 applies to all $(\mathfrak{R}(F), t)$-saturated graphs. Let

$$d = \frac{\log n}{\log \log n}.$$

By Claim 1, G contains at most $t^{d-1}d^d < (td)^d$ vertices of degree at most d. Since t is a constant, we have that

$$(td)^d = \left(t \frac{\log n}{\log \log n}\right)^{\frac{\log n}{\log \log n}} = o(n).$$

It follows that G has $n - o(n)$ vertices of degree at least d, and thus

$$|E(G)| \geq \frac{1}{2}d(n - o(n)) = \Omega\left(n \frac{\log n}{\log \log n}\right).$$

□

We now construct an n-vertex $\mathfrak{R}(F)$-saturated graph with $O(n \log n)$ edges when F is connected and has no independent vertex cut (and so is also 2-connected). In the following, the base of the logarithm is assumed to be 2.

Theorem 4. *Let F be a k-vertex graph that is 2-connected and has no independent vertex cut. For n sufficiently large and $t \geq |E(F)|$,*

$$\operatorname{sat}_t(n, \mathfrak{R}(F)) \leq t(k-2)n \lceil \log n \rceil - \binom{t(k-2)\lceil \log n \rceil + 1}{2}.$$

Proof. We construct an n-vertex $(\mathfrak{R}(F), t)$-saturated graph G with a large (on the order of $n - \log n$) independent set. Let xy be an edge of F such that $d(x) = \delta(F)$. To build G, we begin by building a spanning subgraph G' consisting of many overlapping rainbow copies of $F - xy$.

For $i \in [t]$ and $j \in [\lceil \log n \rceil]$, let $S_{i,j}$ be the $(k-2)$-vertex set $\{v_{i,j}^1, \ldots, v_{i,j}^{k-2}\}$. For $i \in [t]$, let $S_i = \bigcup_{j=1}^{\lceil \log n \rceil} S_{i,j}$, and let $S = \bigcup_{i=1}^{t} S_i$. Let $R = \{v_1, \ldots, v_{n-t(k-2)\lceil \log n \rceil}\}$. For each $i \in [t]$, we will build a graph G_i on $S_i \cup R$ such that R is an independent set and the addition of an edge in color i joining any two vertices in R will complete a rainbow copy of F.

6

For each $i \in [t]$, let $(F - xy)_i$ be a rainbow-colored copy of $F - xy$ that does not use color i. For each $j \in [\lceil \log n \rceil]$, place a copy of $(F - xy)_i - \{x, y\}$ on the vertex set $S_{i,j}$. To complete G_i we add edges between S_i and R as follows. Assign to each $v_\ell \in R$ a binary string $(b_{\ell,1}, \ldots, b_{\ell,\lceil \log n \rceil})$ of length $\lceil \log n \rceil$ so that the strings are distinct. For each $\ell \in [n - t(k - 2) \lceil \log n \rceil]$, add edges joining v_ℓ to $S_{i,j}$ so that

1. if $b_{\ell,j} = 0$, then $S_{i,j} \cup \{v_\ell\}$ induces $(F - xy)_i - \{y\}$, and

2. if $b_{\ell,j} = 1$, then $S_{i,j} \cup \{v_\ell\}$ induces $(F - xy)_i - \{x\}$.

That is, if $b_{\ell,j} = 0$, then v_ℓ plays the role of x with respect to $S_{i,j}$, and if $b_{\ell,j} = 1$, then v_ℓ plays the role of y with respect to $S_{i,j}$.

To complete the construction of G', we take the union of $G_1, \ldots, G_t$. For distinct $v_\ell, v_{\ell'} \in R$, because the binary strings assigned to v_ℓ and $v_{\ell'}$ are distinct, there is a choice of j such that $S_{i,j} \cup \{v_\ell, v_{\ell'}\}$ induces $(F - xy)_i$ for all $i \in [t]$. It follows that the addition of $v_\ell v_{\ell'}$ in any color $i \in [t]$ completes a rainbow copy of F in G_i, and consequently in G'.

We claim that G' does not contain F as a subgraph, regardless of the edge colors. Recall that the edge xy was chosen so that $d(x) = \delta(F)$, and thus $\delta(F - xy) < \delta(F)$. Therefore if a vertex in R were to be in a copy of F, the copy of F would necessarily contain vertices in at least two of the sets $S_{i,j}$. However, the vertices in R form an independent vertex cut that separates $S_{i,j}$ and $S_{i',j'}$ for all distinct pairs (i, j) and (i', j') in $[t] \times [\lceil \log n \rceil]$. Therefore no vertex in R lies in a copy of F, and it is clear that no $S_{i,j}$ contains a copy of F. Therefore F is not a subgraph of G'.

We know that the addition of any edge in any color joining two vertices in R completes a rainbow copy of F. However, there may be nonadjacent vertices $u, v \in V(G')$, either both in S or with one in S and one in R, and a color $i \in [t]$ such that the addition of uv in color i does not complete a rainbow copy of F. If this is the case, then we iteratively add such (colored) edges until no such nonadjacent vertices exist. When no such nonadjacent pairs exist, we have constructed an $(\mathfrak{R}(F), t)$-saturated graph; this graph is G. Since R is an independent set of size $n - t(k - 2) \lceil \log n \rceil$ in G, it follows that G is a subgraph of the complete split graph $K_{t(k-2)\lceil \log n \rceil} \vee \overline{K}_{n-t(k-2)\lceil \log n \rceil}$, and therefore

$$|E(G)| \le t(k - 2)n \lceil \log n \rceil - \binom{t(k - 2) \lceil \log n \rceil + 1}{2}.$$

□

3 Rainbow Saturation Numbers of Some Specific Graphs

In this section, we determine $\mathrm{sat}_t(n, \mathfrak{R}(F))$ exactly for several classes of graphs and give upper and lower bounds on others. We begin with a surprising result: there are graphs whose rainbow saturation numbers are quadratic in n. As discussed in the introduction, for any graphs $H_1, \ldots, H_t$, $\mathrm{sat}(n, \mathcal{M}(H_1, \ldots, H_t)) = O(n)$. Hence, the existence of graphs whose rainbow saturation numbers are quadratic in n illuminates an interesting and fundamental difference between the monochromatic and rainbow colored saturation problems.

3.1 Stars

Recall that the Kneser graph $K(n, k)$ is the graph with $\binom{n}{k}$ vertices, where each vertex represents a different k-subset of $[n]$, and two vertices are adjacent if their corresponding subsets are disjoint. A *blow up* of a graph is obtained by replacing vertices with independent sets and replacing edges with complete bipartite graphs between the independent sets corresponding to the endpoints of the edge.

Theorem 5. *If $t \geq k$ and $n \geq (k+1)(k-1)/t$, then $\mathrm{sat}_t(n, \mathfrak{R}(K_{1,k})) = (1 + o(1))\frac{(k-1)}{2t}n^2$.*

Proof. The result is immediate for $k = 1$. We may therefore assume that $k \geq 2$, and begin by characterizing $(\mathfrak{R}(K_{1,k}), t)$-saturated graphs. Let G be an n-vertex $(\mathfrak{R}(K_{1,k}), t)$-saturated graph, and observe first that no vertex is incident to edges of k or more colors, since otherwise G contains a rainbow $K_{1,k}$. Second, if v is incident to edges of at most $k - 2$ colors, then v must have degree $n - 1$. To see this, suppose not, and let u be a nonneighbor of v. As u is necessarily adjacent to edges of at least $k - 1$ different colors, choose any edge incident to u, of color c; adding the edge uv in color c cannot create a rainbow $K_{1,k}$ at u, as none previously existed, and also cannot create a rainbow $K_{1,k}$ at v, since v sees at most $k - 1$ colors. This contradicts our assumption that G was $(\mathfrak{R}(K_{1,k}), t)$-saturated. Finally, observe that if v and w both see color i, then v and w must be adjacent; otherwise, adding vw in color i cannot create a rainbow $K_{1,k}$.

By the first two observations, we can partition the vertex set into two parts: A set Q of vertices of degree $n-1$ that see at most $k-2$ colors, and a set A of vertices that see exactly $k - 1$ colors. By the third observation, A can be partitioned into $\binom{t}{k-1}$ (possibly empty) cliques according to the set of colors on the edges incident to each vertex. If two such cliques in A correspond to vertices that are incident to edges of a common color, then the cliques must be completely joined. Note that if two of the cliques in the partition of A correspond to vertices to edges with disjoint sets of colors, then these cliques have no edges between them.

Hence, if we contract each clique in the partition of A to a vertex, then the underlying graph formed by A is an induced subgraph of the complement of the Kneser graph $K(t, k-1)$. Equivalently, the complement of $G[A]$ is a blow up of a subgraph of $K(t, k-1)$.

We verify that such a graph is $(\mathfrak{R}(K_{1,k}), t)$-saturated. Since no vertex sees k colors, there is no rainbow $K_{1,k}$. The only missing edges are between cliques in A that correspond to vertices that are incident to disjoint color sets. If v and w are in different cliques in A and vw is added in color i, then we may assume without loss of generality that v did not previously have a neighbor of color i. Now v is incident to edges with k different colors and hence it is the center of a rainbow $K_{1,k}$.

Having established the structure of $(\mathfrak{R}(K_{1,k}), t)$-saturated graphs, in order to find $\mathrm{sat}_t(n, \mathfrak{R}(K_{1,k}))$, it suffices to minimize the number of edges in any such graph. Observe that this is equivalent to maximizing the number of edges in the complement of G. Since $K(t, k-1)$ does not contain $K_{\lfloor t/(k-1) \rfloor + 1}$, it follows that $\overline{G}$ is $K_{\lfloor t/(k-1) \rfloor + 1}$-free. Therefore, by Turán's Theorem [10], $\overline{G}$ contains at most $\frac{t/(k-1)-1}{t/(k-1)} \binom{n}{2}$ edges. Thus

$$
\begin{aligned}
|E(G)| &= \binom{n}{2} - |E(\overline{G})| \\
&\geq \left(1 - \frac{t/(k-1)-1}{t/(k-1)}\right) \binom{n}{2} \\
&= \frac{k-1}{t} \binom{n}{2}.
\end{aligned}
$$

Therefore $\mathrm{sat}_t(n, \mathfrak{R}(K_{1,k})) \geq \frac{k-1}{t} \binom{n}{2}$.

Now, let G consist of $\lfloor t/(k-1) \rfloor$ copies of $K_{n/\lfloor t/(k-1) \rfloor}$. Edge-color the copies of $K_{n/\lfloor t/(k-1) \rfloor}$ with pairwise disjoint sets of $k-1$ colors so that each vertex is incident to edges with $k-1$ distinct colors. This graph is $(\mathfrak{R}(K_{1,k}), t)$-saturated and has $(1 + o(1))\frac{k-1}{t} \binom{n}{2}$ edges. $\qquad \square$

We note that for specific k, n, and t, the exact value of $\mathrm{sat}_t(n, \mathfrak{R}(K_{1,k}))$ can be obtained from the proof of Theorem 5 and Turán's Theorem. Furthermore, Turán's Theorem also implies that all n-vertex $(\mathfrak{R}(K_{1,k}), t)$-saturated graphs of minimum size correspond to edge-colorings of a unique graph.

3.2   Paths

We next consider the rainbow saturation number for paths. We begin with a general lower bound on $\mathrm{sat}_t(n, \mathfrak{R}(P_k))$, where P_k denotes the path of order k, and then show that this bound is correct for P_4.

Proposition 6. *For all $k \geq 4$,*

$$\operatorname{sat}_t(n, \mathfrak{R}(P_k)) \geq n - 1.$$

Proof. Let G be an $(\mathfrak{R}(P_k), t)$-saturated graph of order n. If no component of G is a tree, then G has at least n edges, so we may assume that some component of G is a tree. It suffices to show that G has exactly one such component, so assume by way of contradiction that T_1 and T_2 are two components of G that are trees.

Let v be a leaf in T_1 with neighbor u, and suppose the edge vu is color i. We claim that the color i cannot appear on any other edge of T_1. Suppose otherwise, that $x \in T_1$ is incident to an edge xy of color i. Adding the edge vx in color i must create a rainbow P_k (call it P) that contains vx. Since the color i can be used only once, P must begin $v, x, w, ...$, where $w \neq y$. Since T_1 is a tree, y cannot appear in P, as $P + xy$ would contain a cycle. Hence replacing vx with yx yields a rainbow P_k in G, a contradiction.

Now let v' be a leaf in T_2 with neighbor u', and assume the edge $v'u'$ is color j. Adding the edge $v'u$ in color j must create a rainbow P_k. This rainbow P_k must consist of $v'u$ together with a rainbow P_{k-1} beginning at u, since color j cannot be used twice. The rainbow P_{k-1} beginning at u cannot contain uv, as T_1 is a tree and $k \geq 4$, but it must contain an edge of color i; otherwise, replacing $v'u$ with vu would yield a rainbow P_k. However, this contradicts the fact that color i cannot appear on any edge other than uv in T_1. Therefore G contains at most one tree component. $\qquad\square$

Corollary 7. *For $t \geq 8$, $\operatorname{sat}_t(n, \mathfrak{R}(P_4)) = n - 1$.*

Proof. The lower bound follows from Proposition 6, so it suffices to construct $(\mathfrak{R}(P_4), t)$-saturated graphs of order n with $n - 1$ edges. When $n \not\equiv 0 \pmod 3$, a collection of $\lfloor \frac{n}{3} \rfloor$ rainbow triangles and an additional vertex or edge, depending on the value of $n \bmod 3$, suffices. For $n \equiv 0 \pmod 3$, let H_n be $K_{1,5} + \left(\frac{n-6}{3} \right) K_3$, colored so that $K_{1,5}$ is rainbow with colors from $[5]$ and each copy of K_3 is rainbow with colors from $[t] - [5]$. $\qquad\square$

4 Upper bounds

In this section we prove upper bounds on rainbow saturation numbers with constructions of $(\mathfrak{R}(F), t)$-saturated graphs for various choices of F. First we prove that the rainbow saturation number for most trees is at most linear in n.

Theorem 8. *Let H be a connected k-vertex graph with $k \geq 5$. If H has a vertex v with $d(v) = 1$ whose neighbor v' does not have degree $k - 1$, there are two vertices u and u' in $V(H) \setminus \{v, v'\}$ that are not adjacent, and $t \geq \binom{k-1}{2}$, then*

$$\mathrm{sat}_t(n, \mathfrak{R}(H)) \leq \left\lceil \frac{n}{k-1} \right\rceil \binom{k-1}{2}.$$

In particular, if T is a tree with at least four vertices that is not a star, then

$$\mathrm{sat}_t(n, \mathfrak{R}(T)) = O(n).$$

Proof. Let G be the n-vertex graph formed by partitioning the vertex set into $\lfloor \frac{n}{k-1} \rfloor$ cliques of size $k - 1$ and one clique containing the remaining $n \mod (k - 1)$ vertices. Rainbow color the edges of each clique. We claim that G is $(\mathfrak{R}(H), t)$-saturated. Suppose that the edge xy is added in color i, where x is in the clique $G_x \subseteq G$ and y is in the clique $G_y \subseteq G$. Assume by symmetry that G_x has $k - 1$ vertices; hence G_x contains a rainbow copy of every $k - 1$-vertex graph. We claim that G_x contains a rainbow copy of $H - v$ avoiding the color i. Set $x = v'$. Since G_x is rainbow-colored, G_x contains at most one edge ab with color i. If $x = a$, then let b be a nonneighbor of v' in H. If $x \neq a, b$, then let $a = u$ and $b = u'$. All other vertices of $H - v$ can be assigned arbitrarily to vertices of G_x, and we obtain the needed rainbow $H - v$ avoiding the color i. Now this together with the edge xy in color i creates a rainbow copy of H.

Finally, if T is a tree that is not a star, then either T has order at least 5 or $T = P_4$. Therefore, this construction and Proposition 6 together imply that $\mathrm{sat}_t(n, \mathfrak{R}(T)) = O(n)$. $\square$

Our next construction also provides a linear bound on rainbow saturation numbers for many graphs. This construction is a natural analogue to the construction used for Theorem 4, and it too applies to graphs that have no independent vertex cut. However, it also applies to graphs with girth at least 4 in which some edge has a special relationship with the cycles of minimum or near-minimum length. In particular, graphs of this type include all cycles of length at least 4.

Theorem 9. *Let F be a k-vertex connected graph satisfying one of the following properties:*

1. *there is an edge in F that does not lie on a triangle, F is 2-connected, and F does not have an independent vertex cut;*

2. *F has finite girth at least 4, (call the girth g), and there is an edge xy in F such that more than half of the cycles of length g in F contain xy, any cycle of length g passing through x or y must contain xy, and no cycle of length $g + 1$ contains xy.*

11

It follows that $\mathrm{sat}_t(n, \mathfrak{R}(F)) \leq 2(k-2)n - \binom{2k-3}{2}$. *Therefore* $\mathrm{sat}_t(n, \mathfrak{R}(F)) = O(n)$.

Proof. We describe an n-vertex graph G with $O(n)$ edges that is $(\mathfrak{R}(F), t)$-saturated. We begin by describing a subgraph G' of G. If F satisfies Condition 1 of the statement of the theorem, let xy be an edge in F that does not lie on a triangle. If F satisfies Condition 2, let xy be an edge as described in the condition.

To create G', let F_1 and F_2 be two disjoint copies of $F - xy$. If $u \in V(F)$, then denote by u_i the copy of u in F_i. Let c be a rainbow coloring of the edges of $F_1 \cup F_2$ so that no color is reused. Create an independent set of size $n - 2k$, and let R be that independent set together with $x_1, x_2, y_1,$ and y_2. For every neighbor a_i of x_i in F_i, add an edge of color $c(x_i a_i)$ connecting a_i to each vertex in R. Similarly, for every neighbor b_i of y_i in F_i, add an edge of color $c(v_i y_i)$ connecting y_i to each vertex in R. Since x and y have no common neighbors, this coloring is well defined.

Notice first that adding an edge in any color between vertices of R will create a rainbow copy of F. If the edge uv is added to R in color j, since the color sets of F_1 and F_2 are disjoint, then $(V(F_i) \cup \{u, v\}) - \{x_i, y_i\}$ is the vertex set of a rainbow copy of F for some $i \in \{1, 2\}$.

We claim that G' contains no rainbow copy of F. Observe that the edges of $G'[V(F_i) \cup R]$ have only $|E(F)| - 1$ colors for $i \in \{1, 2\}$. Therefore, any rainbow copy of F in G' would have to contain vertices from both $F_1 - \{x_1, y_1\}$ and $F_2 - \{x_2, y_2\}$. However, if F satisfies Condition 1 of the theorem, then such a copy of F would contain an independent vertex cut in R.

Now suppose that F has girth g and satisfies Condition 2 of the theorem. We will show that G' contains fewer rainbow cycles of length g than F has (uncolored) cycles of that length. Every rainbow path in G' with endpoints in R and internal vertices in $F_i - \{x_i, y_i\}$ corresponds either to a path in $F - xy$ joining x and y, or to a cycle in $F - xy$ that contains x or y. Therefore every rainbow path in G' with endpoints in R contains at least $g - 1$ edges, and any rainbow cycle containing vertices in both $V(F_1)$ and $V(F_2)$ contains at least $2(g-1)$ edges. Therefore each rainbow cycle of length g in G' lies in $G'[V(F_i) \cup R]$ for some $i \in \{1, 2\}$.

We claim that no such rainbow cycle contains any vertex of R. Indeed, if some rainbow cycle C of length g did contain a vertex from R, by symmetry we may assume that the vertex is x_1, and its neighbors on the cycle belong to F_1, with at least one of the neighbors corresponding to a neighbor of x_1 in F_1. As previously noted, after leaving x_1 the cycle passes through at least $g - 1$ edges before encountering another vertex of R; since R is an

independent set and C has length g, the vertex x_1 is the only vertex from R contained in C. The rainbow cycle then corresponds to a cycle containing x in F; if both neighbors of x_1 in C correspond to neighbors of x in F, then F has a cycle of length g that does not contain y, and if instead one of the neighbors of x_1 in C corresponds to a neighbor of y in F, then F has a cycle of length $g + 1$ containing the edge xy. Both possibilities contradict Condition 2, so no rainbow cycle of length g in G' contains any vertex of R.

Thus every rainbow cycle of length g in G' corresponds to a cycle of length g in F_i (for some $i \in \{1, 2\}$) that does not contain the edge $x_i y_i$. Since xy lies on more than half of the cycles of length g in F, it follows the number of rainbow cycles of length g in G' is smaller than the number of cycles of length g in F. Therefore no rainbow copy of F appears in G'.

Note that G' may not be $(\mathfrak{R}(F), t)$-saturated, since there may be nonadjacent vertices $u, v \in V(G')$, either both in $V(G') \setminus R$ or with one in $V(G) \setminus R$ and one in R, and a color $i \in [t]$ such that the addition of uv in color i does not complete a rainbow copy of F. If this is the case, then we iteratively add such (colored) edges until no such nonadjacent vertices exist. When no such nonadjacent pairs exist, we have constructed an $(\mathfrak{R}(F), t)$-saturated graph; this graph is G. Since R is an independent set of size $n - 2(k - 2)$ in G, it follows that G is a subgraph of the complete split graph $K_{2(k-2)} \vee \overline{K}_{n-2(k-2)}$, and therefore

$$|E(G)| \leq 2(k-2)n - \binom{2k-3}{2}. \qquad \square$$

5 Conclusions and Open Problems

There are many open questions remaining regarding rainbow saturation numbers of graphs. We conjecture that the upper bound given by Theorem 4 for complete graphs is the correct order of growth for the rainbow saturation number of complete graphs.

Conjecture 2. $\mathrm{sat}_t(n, \mathfrak{R}(K_k)) = \Theta(n \log n)$.

The quadratic value of $\mathrm{sat}_t(n, \mathfrak{R}(K_{1,k}))$ from Theorem 5 depends strongly on the fact that adding an edge to an $(\mathfrak{R}(K_{1,k}), t)$-saturated graph must increase the number of colors on the edges incident to some vertex in the graph. The next question asks if this is the only property that will lead to quadratic rainbow saturation numbers.

Question 1. Is there a graph G that is not a star such that $\mathrm{sat}_t(n, \mathfrak{R}(G)) = \Theta(n^2)$?

Throughout this paper, we have taken the number of colors available to be a sufficiently large number. There is certainly a host of compelling questions to consider when the number of colors available is small. Observe that $\mathrm{sat}_t(n, \mathfrak{R}(F))$ is nonincreasing in t since every

$(\mathfrak{R}(F), t)$-saturated graph is also $(\mathfrak{R}(F), t+1)$-saturated. This motivates the following general question.

Question 2. How does $\mathrm{sat}_t(n, \mathfrak{R}(F))$ grow as t approaches $|E(F)|$ from above?

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