Havel–Hakimi residues of unigraphs

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Abstract

The residue $r(G)$ of a graph G is the number of zeros left after fully reducing the degree sequence of G via the Havel–Hakimi algorithm. The residue is one of the best known lower bounds on the independence number of a graph in terms of the degree sequence. Though this bound may be arbitrarily weak for graphs in general, we show that if G is the unique realization of its degree sequence, then the independence number of G is either $r(G)$ or $r(G) + 1$, and we characterize the unigraphs corresponding to each value.

Keywords: Havel-Hakimi algorithm, residue, unigraph, independence number, canonical decomposition, combinatorial problems

1. Introduction

The *residue* is a parameter of the degree sequence of a graph, computed by successive applications of the Havel–Hakimi reduction step. Favaron et al. [1] showed that the residue of a degree sequence gives a lower bound on the independence number of any graph having that degree sequence. In fact, among functions of degree sequences, the residue is one of the best lower bounds currently known (see [5]). Its precision is limited, however, by the fact that it is determined solely by the degree sequence, while distinct realizations of a degree sequence may have different independence numbers. In this paper we study the residue and independence number of *unigraphs*, graphs that are the unique realizations of their respective degree sequences. We show that the independence number of a unigraph can exceed its residue by at most 1, and most often the two values are the same. As we do so, we show how the residue of a graph behaves nicely with respect to the canonical decomposition of a graph, as defined in [9].

In this paper all graphs are finite and simple. We denote the degree sequence of a graph G by $d(G)$ and, unless otherwise stated, order its terms $(d_1, \dots, d_n)$ so that $d_1 \geq \cdots \geq d_n$. We use superscripts to denote multiple terms with

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the same value; for example, $(3,3,2,2,2,2)$ will be written as $(3^2,2^4)$. A list d of nonnegative integers is *graphic* if it is the degree sequence of some graph, and a graph G is a *realization* of d if $d(G) = d$. We use $\alpha(G)$ to denote the independence number of G.

The Havel–Hakimi reduction process was introduced in [3, 4] as a means of determining whether or not a list $d = (d_1, \ldots, d_n)$ of nonnegative integers with even sum is graphic. A *reduction step* on d removes the first term d_1, subtracts 1 from the d_1 largest remaining terms, and reorders the terms, if necessary, into descending order. We let d^1 denote the list obtained via a reduction step on d. The Havel–Hakimi Theorem characterizes graphic integer lists in terms of reduction steps.

Havel–Hakimi Theorem. *The list d is graphic if and only if d^1 is graphic.*

As a corollary we see that d is graphic if and only if it can be reduced to a list of zeros by successive reduction steps. If s is the smallest number of reduction steps necessary to reduce d to a list of zeros, then for $i \in \{0, \ldots, s\}$ we let d^i denote the list resulting from i successive reduction steps on d. The *residue* of d, denoted $r(d)$, is the number of zeros in d^s; alternatively, $r(d) = n - s$. The *residue of a graph G*, denoted $r(G)$, is defined by $r(G) = r(d(G))$.

Most proofs of the Havel–Hakimi Theorem (see [14], for example) proceed inductively and actually prove the following.

Proposition 1.1. *If d is a graphic integer list and $d^1, \ldots, d^s$ are the Havel–Hakimi reductions of d, then there exists a realization G of d and vertices $v_1, \ldots, v_s$ in G such that $\deg(G - v_1 - \cdots - v_i) = d^i$ for all $i \in \{1, \ldots, s\}$.*

For the realization G in Proposition 1.1, $r(G)$ gives a lower bound on $\alpha(G)$. Perhaps surprisingly, this relationship holds true for all realizations of d.

Theorem 1.2 ([1]; see also [2] and [7])**.** *For any graph G, the residue is at most the independence number, that is, $r(G) \le \alpha(G)$.*

For some classes of graphs, this bound is sharp. For instance, the following result is an easy exercise in induction.

Observation 1.3. *A graph has residue 1 if and only if it is a complete graph.*

The bound in Theorem 1.2 can also be arbitrarily weak. For example, for $k \ge 1$ the list $d = \left(k^{2k}\right)$ has residue 2, though the complete bipartite graph $K_{k,k}$ is a realization of d with independence number k. Note, however, that d also has a realization G formed by adding the edges of a matching of size k to $2K_k$; this realization satisfies $r(G) = \alpha(G)$. Thus in evaluating the value of the bound in Theorem 1.2, we are led to the following question.

Question 1. *If d is a graphic list, then how big can the difference be between $r(d)$ and $\min\{\alpha(G) : G$ is a realization of $d\}$?*

It is known that the difference in Question 1 can exceed zero; the list $(4^8, 2)$, for instance, has residue 2, though it can be verified directly that every realization has independence number at least 3. On the other hand, the difference can be bounded in some cases, as a theorem of Nelson and Radcliffe [6] shows.

Theorem 1.4 ([6])**.** *If d is a graphic list that is semi-regular (i.e., its maximum and minimum values differ by at most 1), then d has a realization G such that $r(d) \leq \alpha(G) \leq r(d) + 1$.*

As we saw above, if a degree sequence such as (k^{2k}) has realizations with widely varying independence numbers, then its residue will differ greatly from $\alpha(G)$ for some realization G. How good is the residue as an estimator for α, though, when α cannot differ much among realizations of d?

In this paper we approach this question by limiting the number of realizations of d. We prove the following.

Theorem 1.5. *If G is a unigraph, then $r(G) \leq \alpha(G) \leq r(G) + 1$.*

The paper is organized as follows. In Section 2 we give a bound on the residue that applies to all graphic lists and use it to show that $r(G) = \alpha(G)$ for any split graph G. In Section 3 we show how the Havel–Hakimi reduction process interacts with the the canonical decomposition of a graph, as defined by Tyshkevich [8, 9], and reduce the problem of finding the residue of a graph or graphic list to finding the residue of its "core." We conclude in Section 4 with a proof of Theorem 1.5.

2. A bound on the residue

For any graphic list $d = (d_1, \ldots, d_n)$, define $q(d)$ to be $\max\{k : d_k \geq k\}$ if $d_1 > 0$ and 0 otherwise. Our first result in this section relates $r(d)$ and independence numbers of realizations of d to $q(d)$.

Lemma 2.1. *If G is any n-vertex graph with degree sequence d, then $r(G) \leq \alpha(G) \leq n - q(d)$.*

Proof. The first inequality holds by Theorem 1.2. Let W be a subset of $V(G)$ containing $q(d)$ vertices of highest degree in G. If S is any independent set containing a vertex w of W, then $|S| \leq n - d(w) \leq n - q(d)$, so an independent set of size greater than $n - q(d)$ can contain no vertex from W. However, since there are only $n - q(d)$ vertices in $V(G) \setminus W$, the proof is complete. □

A graph is a *split graph* if its vertex set can be partitioned into disjoint sets A and B such that A is an independent set and B is a clique. As we will show, the bounds in Lemma 2.1 hold with equality for split graphs.

If for a graph G there is a list of vertices $v_1, \ldots, v_s$ in G such that $\deg(G - v_1 - \cdots - v_i) = d^i$ for all $i \in \{1, \ldots, s\}$, and d^s is a list of $r(d(G))$ zeros, we say that $v_1, \ldots, v_s$ is a *list of reducing vertices* for G. Proposition 1.1 implies that for every graphic list there is a realization having a list of reducing vertices.

Proposition 2.2. *If G is an n-vertex split graph with degree sequence d, then $r(d) = \alpha(G) = n - q(d)$.*

Proof. By Lemma 2.1, to prove the equalities it suffices to prove that $n - q(d) \leq r(d)$.

Let $d = (d_1, \ldots, d_n)$. Choose a partition A, B of $V(G)$ into an independent set and a clique, respectively, so that $|A|$ is maximized; this implies that every vertex in B has a neighbor in A. By Proposition 1.1, we may assume that G has a list S of reducing vertices; let $v_1, \ldots, v_s$ be the vertices of this list, chosen so as to maximize $|S \cap B|$.

If $S \not\subseteq B$, then we may let i be the smallest index such that $v_i \in A$. Let $G' = G - v_1 - \ldots v_{i-1}$ and $B' = B - v_1 - \cdots - v_{i-1}$. By definition v_i must be a vertex of maximum degree in G', so its degree must be at least as big as that of a vertex in B' with highest degree. Furthermore, since G' is split it is easy to show that the degree of any vertex in B' cannot be smaller than the degree of any vertex in A. It follows that v_i has the same degree as every vertex in B', and these degrees are all equal to $|B'|$. It follows further that $B' \cup \{v_i\}$ is a clique, and $G' \cong K_{|B'|+1} + cK_1$ for some nonnegative integer c. Deleting all v_j such that $j \geq i$ from G' leaves a collection of isolated vertices, including a vertex w from B'. One easily sees, however, that replacing v_i by w in the list $v_1, \ldots, v_s$ gives a list of reducing vertices for G that has a larger intersection with B, a contradiction. Thus $S \subseteq B$.

Since every vertex in B has a neighbor in A, we have $d_G(v_s) \geq |B| \geq |S| = s$, so $q(d) \geq s$. This implies that $n - q(d) \leq n - s = r(d)$, so the proof is complete. $\square$

3. Canonical decomposition and residues

In preparation for our proof of Theorem 1.5 in the next section, we introduce the canonical decomposition of a graph or graphic list, as defined by Tyshkevich [8, 9]. We show how this canonical decomposition affects the residue of a graph or list.

A *splitted graph* $G(A, B)$ is a split graph G with a specified partition A, B of its vertex set such that A is an independent set and B is a clique. We form the *composition* $G(A, B) \circ H$ of a splitted graph $G(A, B)$ and a graph H by adding to the disjoint union of G and H all edges having an endpoint in B and an endpoint in the vertex set of H. For example, if H and G are paths with 3 and 4 vertices, and A and B are the unordered pairs of the endpoints and midpoints of G, respectively, then $G(A, B) \circ H$ is shown in Figure 1. If L is a list of integers, let $|L|$ denote the number of terms in the list. We denote the degree sequence of a splitted graph by $(d_B; d_A)$, where d_B and d_A are the lists (written in descending order) of vertex degrees of vertices in B and A, respectively. We define the composition $(d_B; d_A) \circ d_C$ of the degree sequences of a splitted graph and of a graph to be the list formed by concatenating d_A, d_B, and d_C, adding $|d_C|$ to the values from d_B and $|d_B|$ to the values from d_C, and putting the result in descending order. Note that $(d_B; d_A) \circ d_C$ is the degree sequence of the composition of a splitted graph with degree sequence $(d_B; d_A)$ and a graph with degree sequence d_C.

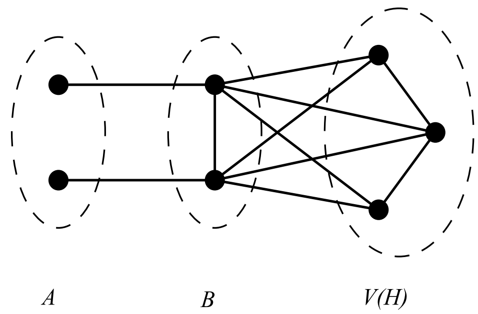

Figure 1: The composition of P_4 and P_3

Note also that in a composition $G(A, B) \circ H$, the degrees of vertices in A are strictly less than the degrees of vertices in the vertex set of H, which are in turn strictly less than the degrees of vertices in B.

A graph is *decomposable* if it can be expressed as the composition of a nonempty split graph and a nonempty graph; it is *indecomposable* otherwise. Tyshkevich showed that every graph has a unique expression as a composition $G_k(A_k, B_k) \circ \cdots \circ G_1(A_1, B_1) \circ G_0$, where each G_i is indecomposable; this is the *canonical decomposition* of the graph, and the graphs G_i are the *canonical components* of G. Note that $G_1, \ldots, G_k$ are split graphs, while the "core" G_0 may not be. Note further that the composition operation $\circ$ is associative, so no grouping parentheses are needed in a canonical decomposition. Decomposable and indecomposable graphic lists and the canonical decomposition of a graphic list are similarly defined, and the analogous uniqueness result holds for graphic lists. Furthermore, a graph is indecomposable if and only if its degree sequence is indecomposable.

We now show how residues interact with the canonical decomposition.

Observation 3.1. *If d is obtained by adding k zeros to the graphic list c, then $r(d) = k + r(c)$.*

Theorem 3.2. *If the graphic list d can be expressed as the composition $d = (d_B; d_A) \circ d_C$, where $(d_B; d_A)$ is indecomposable and both $(d_B; d_A)$ and d_C are nonempty, then $r(d) = |d_A| + r(d_C)$.*

Proof. Suppose d has Havel–Hakimi reductions $d^1, \ldots, d^s$. Let $G = G_1(A, B) \circ G[C]$ be a realization of d with a list of reducing vertices $v_1, \ldots, v_s$. Denote this set of vertices by S. We claim that $|S| \geq |d_B|$ and $\{v_1, \ldots, v_{|d_B|}\} = B$.

Each edge in G must have as an endpoint an element of S, and since $G[B]$ is a complete graph S must contain at least $|B| - 1$ vertices of B; since each vertex in B has a neighbor outside of B, we conclude that $|S| \geq |B| = |d_B|$.

Note that in G the vertices in B have degrees strictly greater than those of vertices in C, which are in turn strictly greater than the degrees of vertices in A. Hence $v_1 \in B$. Indeed, suppose the graph H is formed by deleting from G a

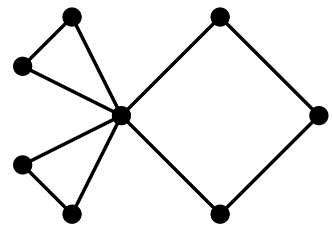

Figure 2: The graph U_2

subset of B of size k, and let B' be the set of vertices remaining in B. During the deletion each vertex of B' has its degree reduced by k, as does each vertex in C. Hence the degrees of vertices B' are still strictly larger than the degrees of vertices in C in H.

Thus the only way one of $v_1, \ldots, v_{|d_B|}$ might not lie in B is if after deleting $v_1, \ldots, v_k$ (with $k < |B|$) some vertex a of A has degree (in the resulting graph H) greater than or equal to the maximum degree of a vertex in B'. Since each vertex in B' is adjacent to every other vertex in B', every vertex in C, and at least one vertex in A, and a only has neighbors in B', we have a contradiction.

Thus $\{v_1, \ldots, v_{|d_B|}\} = B$, and by Observation 3.1,

$$r(d) = r(G) = r(G - v_1 - \cdots - v_{|d_B|}) = |d_A| + r(d_C),$$

as desired. □

Corollary 3.3. *If $G = G_1(A, B) \circ G_0$, where $V(G_0)$ is nonempty (and where neither G_0 nor G_1 need be indecomposable), then $r(G) = |A| + r(G_0)$.*

4. Residues of unigraphs

As stated above, a unigraph is a graph that is the sole realization (up to isomorphism) of its degree sequence. Examples of unigraphs include complete graphs, graphs of the form mK_2, and cycles of length up to 5. In this section we prove Theorem 1.5 by exploiting a characterization of unigraphs due to Tyshkevich and Chernyak [10, 11, 12, 13] (see also [9]). The canonical decomposition is key in this characterization.

Theorem 4.1 ([9])**.** *G is a unigraph if and only if each of its canonical components is a unigraph.*

Thus to characterize unigraphs it suffices to characterize indecomposable unigraphs. For our present purposes, it will suffice to know the nonsplit indecomposable unigraphs. Let U_m denote the graph formed by taking a chordless 4-cycle and m triangles, choosing a vertex in each cycle, and identifying these vertices to form a connected graph. Figure 2 shows the graph U_2.

Theorem 4.2 ([9], Theorem 4)**.** *An indecomposable nonsplit graph H is a unigraph if and only if H or $\overline{H}$ is one of the following: C_5; mK_2 for $m \geq 2$; $mK_2 + K_{1,n}$ for $m \geq 1$ and $n \geq 2$; or U_m for $m \geq 1$.*

We now prove Theorem 1.5, restated in a more precise version.

Theorem 4.3. *If G is a unigraph, then $r(G) \leq \alpha(G) \leq r(G)+1$. Furthermore, if a unigraph G has canonical decomposition $G_k(A_k, B_k) \circ \cdots \circ G_1(A_1, B_1) \circ G_0$, then $\alpha(G) = r(G) + 1$ if and only if $G_0 \cong \overline{U_m}$ for some $m \geq 1$.*

Proof. Theorem 1.2 provides the first inequality.

We claim that in the composition of a splitted graph $H(A, B)$ and a graph J, we have $\alpha(H(A, B) \circ J) = |A| + \alpha(J)$. Indeed, the union of A with a maximum independent set of J gives an independent set of the desired size. However, any independent set larger than this would have to include either more than $\alpha(J)$ vertices of J or at least one vertex of B, which would dominate all vertices of J; either case yields a contradiction. It follows that

$$\alpha(G) = \alpha(G_k(A_k, B_k) \circ \cdots \circ G_1(A_1, B_1) \circ G_0) = \sum_{i=1}^{k} |A_i| + \alpha(G_0).$$

By Corollary 3.3, we likewise have

$$r(G) = r(G_k(A_k, B_k) \circ \cdots \circ G_1(A_1, B_1) \circ G_0) = \sum_{i=1}^{k} |A_i| + r(G_0);$$

hence $\alpha(G) - r(G) = \alpha(G_0) - r(G_0)$.

Hence to complete the proof we need only examine the residue and independence number of G_0, indecomposable and (by Theorem 4.1) a unigraph. If G_0 is split, then $\alpha(G) - r(G) = \alpha(G_0) - r(G_0) = 0$ by Proposition 2.2. Assume henceforth that G_0 is nonsplit. We examine the cases outlined in Theorem 4.2. Let $\omega(H)$ denote the clique number of a graph H.

Case: G_0 is C_5. It is straightforward to check that $r(G_0) = \alpha(G_0) = 2$.

Case: G_0 is mK_2, where $m \geq 2$. One verifies that $\alpha(G_0) = m$. Furthermore, each Havel–Hakimi reduction on $d(G_0)$ deletes one 1 from the list and changes a remaining 1 to a 0, so $r(G_0) = m$ as well.

Case: G_0 is $\overline{mK_2}$, where $m \geq 2$. We have $\alpha(G_0) = \omega(mK_2) = 2$. By Theorem 1.2 and Observation 1.3, $r(G_0) = 2$.

Case: G_0 is $mK_2 + K_{1,n}$, where $m \geq 1$ and $n \geq 2$. In this case $\alpha(G_0) = m+n$, and $d(G_0) = (n, 1^{2m+n})$. One Havel–Hakimi reduction on $d(G_0)$ produces $(1^{2m}, 0^n)$, so $r(G_0) = m + n$.

Case: G_0 is $\overline{mK_2 + K_{1,n}}$, where $m \geq 1$ and $n \geq 2$. Here $\alpha(G_0) = \omega(mK_2 + K_{1,n}) = 2$. By Theorem 1.2 and Observation 1.3, $r(G_0) = 2$ as well.

Case: G_0 is U_m, where $m \geq 1$. By inspection, $\alpha(G_0) = m+2$. Furthermore, $d(G_0) = (2m + 2, 2^{2m+3})$, and if d is this list, then $d^1 = (2, 1^{2m+2}) = d(mK_2 + K_{1,2})$, which we showed above to have residue $m + 2$. Hence $r(G_0) = \alpha(r(G_0))$.

Case: G_0 is $\overline{U_m}$, where $m \geq 1$. We have $\alpha(G_0) = \omega(U_m) = 3$. We now show that $r(\overline{U_m}) = 2$ by induction on m. Note that $d(\overline{U_1}) = (3^5, 1)$; it is easy to verify that the residue is 2. Assume now that $r(\overline{U_{m-1}}) = 2$. If $d = d(\overline{U_m})$, we have

$$\begin{aligned} d &= ((2m+1)^{2m+3}, 1); \\ d^1 &= (2m+1, (2m)^{2m+1}, 1); \\ d^2 &= ((2m-1)^{2m+1}, 1) \\ &= d(\overline{U_{m-1}}), \end{aligned}$$

so $r(G_0) = r(d) = r(d^2) = 2$ by the inductive hypothesis.

We have examined all possible isomorphism classes of G_0 and found that $0 \leq \alpha(G_0) - r(G_0) \leq 1$, with equality in the second inequality if and only if $G_0 \cong \overline{U_m}$ for some $m \geq 1$. This concludes the proof. □

Degree sequences of unigraphs, therefore, join the semi-regular graphic lists as a class of graphic lists in which every element has a realization whose independence number exceeds the residue by at most 1. The author is aware of no graphic list π where every realization of π has independence number at least $r(\pi) + 2$; hence we conclude with the following refinement of Question 1.

Question 2. *Does there exist a constant K such that for every graphic list π there is a realization G of π such that $\alpha(G) \leq r(\pi) + K$?*

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