ON INDUCED SUBGRAPHS, DEGREE SEQUENCES, AND GRAPH STRUCTURE

BY

MICHAEL DAVID BARRUS

B.S., Brigham Young University, 2002
M.S., Brigham Young University, 2004
M.S., University of Illinois at Urbana-Champaign, 2008

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2009

Urbana, Illinois

Doctoral Committee:

Professor Alexandr V. Kostochka, Chair
Professor Douglas B. West, Director of Research
Professor Zoltán Füredi
Assistant Professor József Balogh

ABSTRACT

A major part of graph theory is the study of structural properties of graphs. In this thesis we focus on three topics in structural graph theory that each deal with the set of induced subgraphs of a graph and the degrees of its vertices.

For example, the Graph Reconstruction Conjecture states that any graph on at least three vertices is uniquely determined by the multiset of its unlabeled subgraphs obtained by deleting a single vertex from the graph. This multiset is called the *deck* of the graph, and the induced subgraphs it contains are the *cards*. The *degree-associated reconstruction number* of a graph is the minimum number of cards that suffice to determine the graph when each card is accompanied by the degree of the vertex that was deleted to form it. We obtain results on the degree-associated reconstruction number for graphs in general and for various special classes of graphs, including regular graphs, vertex-transitive graphs, trees, and caterpillars.

Several interesting classes of graphs are characterized by specifying a (possibly infinite) list of *forbidden subgraphs*, that is, graphs that are not allowed to appear as induced subgraphs of graphs in the given class. A number of graph classes have forbidden subgraph characterizations and also have characterizations that rely solely on the degree sequence; examples of such graph classes include the classes of complete graphs and split graphs. We consider the problem of determining which sets $\mathcal{F}$ of forbidden subgraphs are *degree-sequence-forcing*, that is, the set of $\mathcal{F}$-free graphs has a characterization requiring no more information about a

graph than its degree sequence.

Finally, we define the A_4-*structure* H of a graph G to be the 4-uniform hypergraph on the vertex set of G where four vertices comprise an edge in H if and only if they form the vertex set of an alternating 4-cycle in G. Our definition is a variation of the notion of the P_4-structure, a hypergraph which has been shown to have important ties to the various decompositions of a graph. We show that A_4-structure has many properties analogous to those of P_4-structure, including connections to a special type of graph decomposition called the *canonical decomposition*. We also give several equivalent characterizations of the class of A_4-*split graphs*, those having the same A_4-structure as some split graph.

To my wife Michelle.

Acknowledgments

This thesis was made possible through the support of several people. I would like to thank Professors József Balogh, Zoltán Füredi, Alexandr V. Kostochka, and Douglas B. West for serving as my thesis committee and for giving me encouragement as I have begun my research career. I am especially grateful to Professor West for the many hours he has spent helping me as my teacher and advisor. I have gained from him an appreciation for the vast landscape of graph theory. He has also shown me how important good mathematical writing is and how ideas and results can be mulled over, rethought, and reworked to produce an outcome that is elegant and beautiful. I have benefitted from this tutelage several times, and this reworking has made this thesis much better than it began.

I wish also to thank Professor Stephen G. Hartke of the University of Nebraska–Lincoln and Mohit Kumbhat for the hours (and years) we have spent discussing degree-sequence-forcing sets. I have appreciated Mohit's companionship as we have progressed through our graduate program, and Stephen has shared invaluable experience and advice, especially as I have neared the end of that program.

Thanks also go to Professor Wayne W. Barrett of Brigham Young University for introducing me to graph theory and for getting me started on organized mathematical research. Much of this thesis owes its existence to questions and germs of ideas I was first exposed to as Professor Barrett's student.

I would like to thank my parents Michael and Sandy Barrus for their support through my years of schooling. They have always encouraged me to take advantage of opportunities to learn, and without their support I could never have started or continued my education.

Finally, my deepest thanks go to my wife Michelle and our daughters Katy and Abigail. My wife has stood beside me through several trying times in these past years, and I could not have finished graduate school or this thesis without her patience, encouragement, good sense, and love. Thanks, Michelle, for your and the girls' smiles. They have meant everything to me.

TABLE OF CONTENTS

CHAPTER 1

Introduction

A major part of graph theory is the study of structural properties of graphs. (We refer the reader to Section 1.4 at the end of the chapter for basic graph theory definitions and notation.) In this thesis we focus on three topics in structural graph theory that each deal with the set of induced subgraphs of a graph and the degrees of its vertices.

For example, the Graph Reconstruction Conjecture states that any graph on at least three vertices is uniquely determined by the multiset of its unlabeled subgraphs obtained by deleting a single vertex from the graph. This multiset is called the *deck* of the graph, and the induced subgraphs it contains are the *cards*. The *degree-associated reconstruction number* of a graph is the minimum number of cards that suffice to determine the graph when each card is accompanied by the degree of the vertex that was deleted to form it. In Chapter 2 we obtain results on the degree-associated reconstruction number for graphs in general and for various special classes of graphs, including regular graphs, vertex-transitive graphs, trees, and caterpillars. This is joint work with Douglas B. West and appears in [6].

Several interesting classes of graphs are characterized by specifying a (possibly infinite) list of *forbidden subgraphs*, that is, graphs that are not allowed to appear as induced subgraphs of graphs in the given class. A number of graph classes have forbidden subgraph characterizations and also have characterizations that rely solely on the degree sequence; examples of such graph classes include the classes of complete graphs and split graphs. In Chapter 3 we consider the problem of determining which sets $\mathcal{F}$ of forbidden subgraphs are *degree-sequence-forcing*, that is, the set of $\mathcal{F}$-free graphs has a characterization requiring no more information about a graph than its degree sequence. This is joint work with Stephen G. Hartke and Mohit Kumbhat and appears in [3] and [4].

Finally, in Chapter 4 we define the *A_4-structure* H of a graph G to be the 4-uniform hypergraph on the vertex set of G where four vertices comprise an edge in H if and only if they form the vertex set of an alternating 4-cycle in G. Our definition is a variation of the notion of the P_4-structure, a hypergraph which has been shown to have important ties to the various decompositions of a graph. We show that A_4-structure has many properties analogous to those of P_4-structure, including connections to a special type of graph decomposition called the *canonical decomposition.* We also give several equivalent characterizations of the class of *A_4-split graphs*, those having the same A_4-structure as some split graph. This is joint work with Douglas B. West and appears in [5].

1.1 Degree-associated reconstruction numbers

Our first results deal with a problem in graph reconstruction. A *card* of a graph G is an unlabeled subgraph obtained by deleting a single vertex from G. The *deck* of G is the multiset of cards of G. The Graph Reconstruction Conjecture, one of the most prominent unsolved problems in graph theory, is due to Kelly [30] and Ulam [52]. It states that no two nonisomorphic graphs on at least three vertices have identical decks. Results so far have shown how to determine many properties of a graph from its deck, and the conjecture has been proved for various classes of graphs. However, the problem in its full generality remains open at this time.

Motivated by questions on the reconstruction of directed graphs, Ramachandran [44] proposed that the Reconstruction Conjecture be weakened by presenting each vertex-deleted subgraph along with the degree of the deleted vertex in a *degree-associated card*, or *dacard*. Ramachandran's conjecture that each graph is uniquely determined by its *degree-associated deck*, or *dadeck*, is equivalent to the Graph Reconstruction Conjecture for graphs on at least three vertices, since from the entire deck of a graph one can determine the degrees of the deleted vertices. Each dacard, however, gives more information about the graph than the corresponding card does, and dacards provide more information than cards in situations when an entire (da)deck is not specified.

One does not always need the entire deck or degree-associated deck to uniquely reconstruct a graph. Harary and Plantholt [22] defined the *reconstruction number* $\mathrm{rn}(G)$ of a graph G to be the minimum size of a subdeck for which G is the only graph having those cards. Ramachandran [45] modified this definition to define the *degree-associated reconstruction number* $\mathrm{drn}(G)$ of a graph G as the minimum number of dacards that suffice to uniquely determine G. The Reconstruction Conjecture is equivalent to showing that $\mathrm{drn}(G)$ is defined (and at most $|V(G)|$) for each graph G. We observe that $\mathrm{drn}(G) \le 2$ for almost all graphs G (asymptotically), and we show that a graph G satisfies $\mathrm{drn}(G) = 1$ if and only if G or its complement has an isolated vertex or a pendant vertex whose deletion yields a vertex-transitive graph.

We also study the degree-associated reconstruction number for vertex-transitive graphs. These graphs are of interest because they are the graphs for which all dacards are the same. Vertex-transitive graphs are regular; we show that if G is any k-regular graph, then $\mathrm{drn}(G) \le \min\{k+2, n-k+1\}$. We show that $\mathrm{drn}(G) \ge 3$ for every vertex-transitive graph G that is not a complete or edgeless graph. We define a vertex-transitive graph G to be *coherent* if in any two-vertex-deleted subgraph the only way to add a vertex v back to form a card of G is to give v the same neighborhood as one of the deleted vertices. We show that $\mathrm{drn}(G) = 3$

for coherent vertex-transitive graphs, and we show that the Petersen graph, hypercubes, prisms of complete graphs, and disjoint copies of identical coherent graphs are coherent. Nevertheless, we show that vertex-transitive graphs can have large degree-associated reconstruction numbers. Let G be a non-complete vertex-transitive graph in which no two vertices have the same neighborhood. Let $G^{(m)}$ denote the graph obtained by replacing each vertex of G by an independent set of size m and making copies of vertices adjacent in $G^{(m)}$ if the corresponding original vertices are adjacent in G. We prove that $\mathrm{drn}(G^{(m)}) = m + 2$ for any $m \geq 2$.

We also study the degree-associated reconstruction number of trees. Myrvold [41] showed that $\mathrm{rn}(T) \leq 3$ (and hence $\mathrm{drn}(T) \leq 3$) for any tree T other than P_4; we show that $\mathrm{drn}(T) = 2$ when T is a *caterpillar* (a tree that becomes a path when all its leaves are deleted) other than a star or a particular six-vertex tree. The proofs of many of our results make use of the *centroid* of a tree, a notion employed extensively in Myrvold's paper, and we show that $\mathrm{drn}(T) = 2$ for any tree T having exactly one centroid vertex u and a leaf ℓ adjacent to u such that $T - \ell$ also has exactly one centroid vertex.

1.2 Degree-sequence-forcing sets

We next consider the problem of determining which hereditary graph families have characterizations that can be stated strictly in terms of their degree sequences. Such degree sequence characterizations are desirable because conditions depending only on the degree sequence can often be tested by linear-time algorithms. Call a graph family $\mathcal{G}$ *degree-determined* if the question of whether a graph H belongs to $\mathcal{G}$ can be answered knowing only the degree sequence of H. Unfortunately, most graph classes of broad interest are not degree-determined. Some, however, are:

examples include the complete, split, matrogenic, and matroidal graphs (these last two classes will be defined in Chapter 4. Each of these families has a linear-time recognition algorithm based on a degree sequence characterization [20, 50].

A class $\mathcal{G}$ of graphs is *hereditary* if every induced subgraph of an element of $\mathcal{G}$ is also in $\mathcal{G}$. For every hereditary class $\mathcal{G}$ there is a minimal set $\mathcal{F}$ of graphs such that a graph H is in $\mathcal{G}$ if and only if it is $\mathcal{F}$*-free*, that is, it contains no induced subgraph isomorphic to an element of $\mathcal{F}$. We call the elements of $\mathcal{F}$ *forbidden subgraphs* for the class $\mathcal{G}$.

We seek to characterize degree-determined hereditary families by studying their associated minimal forbidden subgraphs. We define a set $\mathcal{F}$ of graphs to be *degree-sequence-forcing* if the class of $\mathcal{F}$-free graphs is degree-determined. We observe that if the $\mathcal{F}$-free graphs are the unique realizations of their respective degree sequences, then $\mathcal{F}$ is degree-sequence-forcing. We show that every degree-sequence-forcing set must contain a disjoint union of complete graphs, a complete multipartite graph, a forest of stars, and the complement of a forest of stars. As a consequence, there are only three singleton sets and eleven pairs of graphs that are *minimal* degree-sequence-forcing sets, meaning that no proper subset is also degree-sequence-forcing.

We also characterize the non-minimal degree-sequence-forcing triples, showing that they all belong to one of ten infinite families or a collection of twenty-seven other sets. In the process, we consider an analogue of degree-sequence-forcing sets for bipartite graphs with a fixed bipartition. We also study minimal degree-sequence-forcing sets, showing that for any natural number k, there are finitely many minimal degree-sequence-forcing k-sets.

For certain hereditary families $\mathcal{G}$, the degree sequence of a graph H determines not only whether H is in $\mathcal{G}$, but also how many edges must be added to or deleted from H to produce a graph in $\mathcal{G}$. When $\mathcal{G}$ is the class of split graphs, this parameter is known as the *splittance* of H; Hammer and Simeone [20] defined the splittance and gave a formula for it in terms of the degree sequence. Degree-determined families having this additional degree sequence property are called *edit-level*. If a hereditary class of graphs is edit-level, then its set of minimal forbidden subgraphs is *edit-leveling*. Edit-leveling sets of graphs are necessarily degree-sequence-forcing, though the converse is not true. We give examples of edit-leveling sets and a show that if $\mathcal{F}$ is edit-leveling, then so is $\mathcal{F}^{(k)}$, the set of minimal forbidden subgraphs for the family of graphs that can be produced by adding or deleting at most k edges from an $\mathcal{F}$-free graph. We prove that a set of graphs is edit-leveling if and only if $\mathcal{F}^{(k)}$ is degree-sequence-forcing for every natural number k.

1.3 The A_4-structure of a graph

In work related to Berge's Strong Perfect Graph Conjecture, Chvátal [13] defined the P_4-*structure* of a graph G as the 4-uniform hypergraph having the same vertex set as G in which four vertices form an edge if and only if they induce a path (a copy of P_4) in G. He conjectured that two graphs having the same P_4-structure are either both perfect or both imperfect (this result, initially called the Semi-Strong Perfect Graph Conjecture, was later proved by Reed [47]). Research on the P_4-structure has since grown beyond a focus on perfect graphs; the P_4-structure has been used to define several graph classes with interesting structural properties in which several optimization problems can be solved more efficiently than on general graphs. It has also appeared in several schemes of graph decomposition (partitioning the vertex set of a graph into subsets with prescribed properties).

An induced P_4 in a graph gives rise to an *alternating 4-cycle*, a configuration on four vertices in which two edges and two non-edges of the graph alternate in a cyclic fashion. We define the A_4-*structure* of a graph G by modifying the definition of the P_4-structure to include as edges the vertex sets of all alternating 4-cycles. We observe that the threshold graphs (which will be defined in Chapter 4) are those whose A_4-structures contain no edges, and the matrogenic and matroidal graphs are those whose A_4-structures contain no five vertices inducing exactly two or three edges, and contain no five vertices containing more than one edge, respectively. We also show that cycles of length 5 or at least 7 are, together with their complements, the unique graphs having their particular A_4-structure. As a consequence, we give an A_4-structure analogue of the Semi-Strong Perfect Graph Theorem. We also prove that triangle-free graphs G and H have the same A_4-structure if and only if there is a bijection $\varphi : V(G) \to V(H)$ such that whenever S is the vertex set of a matching of size at least 2 in G, $\varphi(S)$ is the vertex set of a matching in H.

Several results on P_4-structure have analogues in the context of graph A_4-structures. In particular, a *module* in a graph is a vertex subset S such that no vertex outside S has both a neighbor and a non-neighbor in S. We introduce an analogous concept by defining a *strict module* to be a module S such that no *alternating path* (a configuration to be defined in Chapter 4) begins and ends in S. We show that the relationship between the P_4-structure and the modules of a graph is analogous to the relationship between the A_4-structure and the strict modules of the graph; in particular, 4-vertex graphs having alternating 4-cycles are the minimal graphs not having any nontrivial strict modules.

We show further that strict modules and A_4-structure are closely related to the *canonical decomposition* of a graph, as defined by Tyshkevich [49,51]. In particular, the indecomposable components of a graph under this decomposition have the same vertex sets as the connected components of the A_4-structure of the graph. As a result, the role of the A_4-structure of a graph in its canonical decomposition is an analogue of the role of the P_4-structure in the *primeval decomposition* defined by Jamison and Olariu [29].

Finally, we show that the canonical decomposition of a graph can be used to generate other graphs having the same A_4-structure. Motivated by these results, we define the A_4*-split* graphs, those having the same A_4-structure as some split graph. We give five equivalent characterizations of the A_4-split graphs, including a list of eleven forbidden induced subgraphs and a characterization in terms of the canonical decomposition.

1.4 Definitions and notation

A *graph* G consists of two sets $V(G)$ and $E(G)$ called the *vertex set* and the *edge set* of G, respectively. Each member of $V(G)$ is called a vertex; each member of $E(G)$ is an unordered pair of distinct vertices, called an edge. The *order* of a graph is the size of its vertex set. All graphs in this thesis are assumed to have positive, finite order.

In writing edges of a graph, we use uv to denote an edge $\{u, v\}$, and we refer to u and v as the *endpoints* of the edge. If uv is an edge, then u and v are *adjacent*, and the edge uv is *incident with* u and v. The *neighborhood* $N_G(v)$ of a vertex v in G is the set of all vertices adjacent to v; these are the *neighbors* of v. The *closed neighborhood* $N_G[v]$ is $N_G(v) \cup \{v\}$.

An *ismorphism* from a graph G to a graph H is a bijection $\varphi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(H)$. If such an isomorphism exists, we say that G and H are *isomorphic* and denote this by $G \cong H$. An isomorphism from G to itself is an *automorphism* of G. A graph G is *vertex-transitive* if for every pair (u, v) of vertices in G there exists an automorphism of G mapping u to v.

Graph isomorphism defines an equivalence relation on graphs, and the equivalence class containing a graph G is the *isomorphism class* of G. Often we will refer to an isomorphism class as a single graph, such as when we speak of *the* graph on n vertices with no edges. When we wish to emphasize that an isomorphism class of graphs is meant, rather than a single member of the isomorphism class, we use the term *unlabeled graph.* A graph G is a *copy of* a graph or isomorphism class if G is isomorphic to the graph or belongs to the isomorphism class in question.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is (isomorphic to) a subgraph of G, then we may say that G *contains* H or that H *is contained in* G. The graph H is an *induced subgraph* of G if H is a subgraph of G with the property that two vertices are adjacent in H if and only if they are adjacent in G. If H is (isomorphic to) an induced subgraph of G, we say that H *is induced in* G, or that G *induces* (a copy of) H. If $S \subseteq V(G)$, the subgraph *induced by* S, denoted $G[S]$, is the induced subgraph of G having vertex set S.

To *delete* a vertex v from a graph G is to remove v from $V(G)$ and to remove from $E(G)$ all edges containing v. The resulting graph equals $G[V(G) - \{v\}]$ and is denoted by $G - v$. We may denote the result of deleting all vertices in a set S from G by $G - S$. The graph H is induced in G if and only if H may be obtained by deleting vertices from G. To delete an edge uv from a graph G is to remove uv from $E(G)$; we denote the resulting graph by $G - uv$.

The *complement* $\overline{G}$ of a graph G is the graph with vertex set $V(G)$ where any two vertices are adjacent if and only if they are not adjacent in G.

The *degree* of a vertex v in a graph G is the number of edges of G incident with v. We denote the degree of v in G by $d_G(v)$, or simply $d(v)$ when G is understood. The maximum and minimum vertex degrees in G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. If $\Delta(G) = \delta(G)$, then G is *regular*; if $d_G(v) = k$ for all $v \in V(G)$, then G is *k-regular*. A vertex of degree 0 is an *isolated* vertex, a vertex of degree 1 is a *leaf* or *pendant vertex*, and a vertex in G whose degree is $|V(G)| - 1$ (that is, the vertex is adjacent to all other vertices in G) is a *dominating vertex*. Given a vertex subset S and a vertex v, we say that v *is isolated from* S if v is adjacent to no vertex of S, and v *dominates* S if $S \subseteq N_G(v)$.

The list of vertex degrees in an n-vertex graph is the *degree sequence*, and it is usually written as an n-tuple with its entries in nonincreasing order. If a graph G has degree sequence d, then G is a *realization of* d. If G is a realization of $(d_1, \ldots, d_n)$ and m is the number of edges in G, then since each edge is incident with its two endpoints, we have the well-known *Degree-Sum Formula*, which states that

$$\sum_{i=1}^{n} d_i = 2m.$$

An *independent set* is a set of vertices that are pairwise nonadjacent. A *clique* is a set of vertices that are pairwise adjacent. A *k-clique* is a clique of size k.

The *chromatic number* of a graph G is the smallest number of independent sets that together partition $V(G)$. A graph G is *perfect* if for every induced subgraph G' of G the chromatic number of G' equals the maximum size of a clique in G'.

A *path* on n vertices, denoted P_n, is a graph whose vertex set may be indexed $\{v_1, \ldots, v_n\}$ so that its edge set is $\{v_i v_{i+1} : 1 \leq i \leq n-1\}$. We denote such a path in Chapters 1–3 by $\langle v_1, v_2, \ldots, v_n \rangle$. (In Chapter 4 our focus will be on *alternating* paths, and we will redefine this notation then.) The first and last vertices are the *endpoints* of the path, and the remaining vertices are the *interior vertices*. The *length* of the path is the number of edges it contains. A *cycle* on n vertices, also called an *n-cycle* and denoted C_n, is a graph formed by adding an edge joining the endpoints of a path on n vertices. We denote a cycle with vertices $v_1, \ldots, v_n$ in cyclic order by $[v_1, v_2, \ldots, v_n]$. A *triangle* is a graph isomorphic to C_3.

A graph is *connected* if for any two vertices u and v in the graph, there is a path having u and v as its endpoints; it is *disconnected* otherwise. A *component* in a graph is a maximal connected subgraph. A *cut-vertex* in a graph is a vertex whose deletion leaves the resulting graph with more components than the original graph had; a *cut-edge* is an edge having the same property.

The *distance* between two vertices u and v in G is the number of edges on a shortest path having endpoints u and v. We write $\mathrm{diam}(G)$ for the *diameter* of G, which is the largest distance between vertices in G.

A *tree* is a connected graph with no cycles. A *forest* is a graph in which every component is a tree. A tree on n vertices has exactly $n-1$ edges. A graph G is *bipartite* if its vertex set may be partitioned into two sets A and B, called the *partite sets*, such that A and B are independent sets in G. A *matching* in G is a set of pairwise disjoint edges.

A graph G is *edgeless* if $V(G)$ is an independent set. A graph is *complete* if its vertex set is a clique. We use K_n to denote the complete graph on n vertices, and we denote by $K_n - e$ the unlabeled graph obtained by deleting any edge of K_n. A *complete multipartite graph*, denoted $K_{n_1,\ldots,n_k}$, is a graph whose vertex set may be partitioned into subsets $V_1, \ldots, V_k$ (the *partite sets*) with orders $n_1, \ldots, n_k$, respectively, such that vertices $u \in V_i$ and $v \in V_j$ are adjacent if and only if $i \neq j$. If $k = 2$, we refer to the graph as a *complete bipartite graph.* A *star* is a graph of the form $K_{1,m}$; equivalently, it is a tree with diameter at most 2. A graph is a *split graph* if its vertex set can be partitioned into a clique and an independent set.

The *disjoint union* of graphs G and H, denoted $G + H$, is the graph whose vertex set is $V(G) \cup V(H)$ and whose edge set is $E(G) \cup E(H)$, where we assume that G and H have disjoint vertex sets and disjoint edge sets. When a disjoint union is taken of a graph with itself, we denote the result with a coefficient; the graph $G + G + \cdots + G$ (m copies) is denoted mG. The *join* of disjoint graphs G and H, denoted $G \vee H$, is the graph whose vertex set is $V(G) \cup V(H)$ and whose edge set is $E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

The *cartesian product* $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ such that (u, v) and (u', v') are adjacent precisely when (i) $u = u'$ and $vv' \in E(H)$ or (ii) $v = v'$ and $uu' \in E(G)$. When $H = K_2$, the special case $G \square K_2$ of the cartesian product is formed from $2G$ by adding a matching of size $|V(G)|$ joining the two copies of each vertex of G; this is the *prism over* G.

A *hypergraph* is a pair (V, E) where the set V contains *vertices* of the hypergraph, and the set E contains subsets of V of any size (as opposed to graphs). A hypergraph is k-uniform if every edge contains exactly k vertices. A hypergraph H is *connected* if for every two vertices u and v in H there is a list $u_1, \ldots, u_k$ of vertices such that $u_1 = u$ and $u_k = v$ and every two consecutive vertices in the list belong to an edge of H. If the shortest such list has length ℓ, then the *distance* between u and v is $\ell - 1$. The maximal connected subhypergraphs of a hypergraph are its *components*.

A hypergraph H' is a *subhypergraph* of H if $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H))$. Given hypergraphs H and J, a *hypergraph isomorphism from H to J* is a map $\varphi : V(H) \to V(J)$ such that for every subset A of $V(H)$, we have $\varphi(A) \in E(J)$ if and only if $A \in E(H)$.

CHAPTER 2

Degree-associated reconstruction numbers

2.1 Introduction

The well-known Graph Reconstruction Conjecture of Kelly [30] and Ulam [52] has been open for more than 50 years. It asserts that every graph with at least three vertices can be (uniquely) reconstructed from its "deck" of vertex-deleted subgraphs. Here the *deck* of a graph G is the multiset of unlabeled induced subgraphs formed by deleting one vertex from G, and these subgraphs are *cards* in the deck. Saying that G is reconstructible is the same as saying that all graphs with the same deck as G are isomorphic to G. The conjecture has been proved for many special classes of G, and many results show that various properties of G may be deduced from its deck. Nevertheless, the full conjecture remains open. Surveys of results on reconstruction include [9, 10, 33, 35].

It may not be necessary to know the entire deck to reconstruct the graph. Harary and Plantholt [22] defined the *reconstruction number* of a graph G, denoted $\mathrm{rn}(G)$, to be the minimum number of cards from the deck that suffice to determine G. The Reconstruction Conjecture is the statement that $\mathrm{rn}(G)$ is defined (at most $|V(G)|$) for each graph G with at least three vertices. Reconstruction numbers are known for various classes of graphs; see [1, 22, 38–41].

Motivated by reconstruction questions for directed graphs, Ramachandran [44] proposed a slightly different model. A *degree-associated card* (or *dacard*) of a graph (or digraph) is a pair (C, d) consisting of a card C in the deck and the degree (or in/out-degree pair) d of the deleted vertex. The multiset of dacards is the *dadeck* (the *degree-associated deck*). For graphs with at least three vertices, knowing the degree of the deleted vertex is equivalent to knowing the total number of edges. A simple counting argument computes $|E(G)|$ when the entire deck is known, so the dadeck gives the same information as the deck. However, the counting argument requires the entire deck, so an individual dacard gives more information than the corresponding card. Ramachandran [45] defined the *degree-associated reconstruction number* $\mathrm{drn}(G)$ of a graph G to be the minimum number of dacards that suffice to determine G. Clearly $\mathrm{drn}(G) \leq \mathrm{rn}(G)$. Ramachandran studied this parameter for complete graphs, edgeless graphs, cycles, complete bipartite graphs, and disjoint unions of identical graphs.

In this chapter we continue this study. Bollobás [8] proved that $\mathrm{rn}(G) = 3$ for almost every graph. In Section 2.2 we conclude from this that $\mathrm{drn}(G) \leq 2$ for almost every graph, and we characterize the graphs G for which $\mathrm{drn}(G) = 1$. We also prove that $\mathrm{drn}(G) \leq \min\{k+2, n-k+1\}$ when G is a k-regular graph with n vertices.

In Section 2.3 we study vertex-transitive graphs. Let G be vertex-transitive. We prove that $\mathrm{drn}(G) \geq 3$ and give a sufficient condition for equality; it holds for the Petersen graph, the k-dimensional hypercube, and the cartesian product $K_n \square K_2$. Also, if G has nonadjacent vertices with distinct neighborhoods, and $G^{(m)}$ arises from G by expanding each vertex into m independent vertices, then $\mathrm{drn}(G^{(m)}) = tm + 2$, where t is the maximum number of vertices having the same neighborhood in G.

In Sections 2.4–2.6 we study trees. Section 2.4 gives sufficient conditions for $\mathrm{drn}(G) = 2$ when G is a tree. These aid subsequently in computing the value for all trees whose non-leaf vertices form a path; these trees are called *caterpillars*. If G is a caterpillar, then $\mathrm{drn}(G) = 2$ unless G is a star or the one 6-vertex tree with four leaves and maximum degree 3. We consider special families of caterpillars in Section 2.5 and complete the general proof in Section 2.6.

2.2 Small reconstruction numbers and regular graphs

In this section we show that $\mathrm{drn}(G) \leq 2$ for almost every graph G, and we determine when $\mathrm{drn}(G) = 1$. Our observation relies heavily on the result of Bollobás [8] about $\mathrm{rn}(G)$, which also implies that almost every graph is reconstructible.

Theorem 2.1 (Bollobás [8])**.** *Almost every graph has reconstruction number 3. Furthermore, for almost every graph, any two cards in the deck determine everything about the graph except whether the two deleted vertices are adjacent.*

The reconstruction number of any graph is at least 3, since $G-u$ and $G-v$ are cards for both G and G', where G and G' differ only on whether the edge uv is present. Thus, the previous result is sharp. The degree information determines the last unknown bit of information without introducing another card.

Corollary 2.2. *For almost every graph G,* $\mathrm{drn}(G) \leq 2$.

Proof. Let G be a graph with two cards that determine the graph except for whether the deleted vertices are adjacent. In the dadeck of G the cards $G-u$ and $G-v$ are paired with $d_G(u)$ and $d_G(v)$. The degree information determines whether uv is present, thereby reconstructing G; thus $\mathrm{drn}(G) \leq 2$. □

It is natural to ask when $\mathrm{drn}(G) = 1$. We answer this question in the next few results.

Lemma 2.3. *For any graph G,* $\mathrm{drn}(G) = \mathrm{drn}(\overline{G})$.

Proof. Let v be a vertex in an n-vertex graph G. Since $d_{\overline{G}}(v) = n-1-d_G(v)$ and $\overline{G-v} = \overline{G}-v$, it follows that (C,d) is a dacard of G if and only if $(\overline{C}, n-1-d)$ is a dacard of $\overline{G}$.

Consider a multiset $\{(C_1, d_1), \ldots, (C_r, d_r)\}$ of dacards that determine G. Since these can be obtained from $\{(\overline{C}_1, n-1-d_1), \ldots, (\overline{C}_r, n-1-d_r)\}$ and $\overline{G}$ can be obtained from G, we conclude that $\mathrm{drn}(\overline{G}) \leq \mathrm{drn}(G)$. Reversing the roles of G and $\overline{G}$ yields $\mathrm{drn}(G) = \mathrm{drn}(\overline{G})$. □

Note that $\mathrm{drn}(G) = 1$ if and only if G has a dacard that does not occur in the dadeck of any other graph. We next determine all dacards of this type.

Theorem 2.4. *The dacard (C, d) belongs to the dadeck of only one graph (up to isomorphism) if and only if one of the following holds:*

(1) $d = 0$ *or* $d = |V(C)|$*;*

(2) $d = 1$ *or* $d = |V(C)| - 1$*, and* C *is vertex-transitive;*

(3) C *is complete or edgeless.*

Proof. Let $n = |V(C)|$. In each case listed, there is exactly one way (up to isomorphism) to form a graph G with $n+1$ vertices by adding to C a vertex with d neighbors in C.

Suppose now that (C, d) is a dacard for only one graph. That is, adding a vertex adjacent to d vertices in C produces a graph in the same isomorphism class no matter which d vertices of C are chosen. If (C, d) is not in the list above, then $d \notin \{0, n\}$ and $C \notin \{K_n, \overline{K}_n\}$. We must show that then $d \in \{1, n-1\}$ and C is vertex-transtive.

Because (C, d) is a dacard for only one graph, the same isomorphism class is produced no matter what set of d vertices is chosen for the neighborhood of the added vertex v. Since isomorphic graphs have the same number of triangles, and the number of triangles after adding v is the number of triangles in C plus the number of edges in C induced by neighbors of v, we conclude that every induced subgraph of C with d vertices has the same number of edges. It is a well-known exercise (Exercise 1.3.35 on page 50 of [53]) that when $1 < d < n-1$, this property forces $C \in \{K_n, \overline{K}_n\}$.

Hence we may assume that $d \in \{1, n-1\}$. Since (C, d) determines G if and only if $(\overline{C}, n-1-d)$ determines $\overline{G}$, we many assume that $d = 1$. Note that adding a vertex of degree 1 adds 1 to some vertex degree in C. In particular, (C, d) is a dacard for some graph with maximum degree $\Delta(C) + 1$. If C is not regular, then also (C, d) is a dacard for some graph with maximum degree $\Delta(C)$. Hence C must be regular.

If C is regular of degree 0 or 1, then automatically C is vertex-transitive. For larger degree, every automorphism of the resulting graph G fixes v, since it is the only vertex of degree 1. Since attaching v to any vertex yields the same graph, C must have automorphisms taking each vertex to any other. Hence C is vertex-transitive. □

Corollary 2.5. *A graph G satisfies* $\text{drn}(G) = 1$ *if and only if G or $\overline{G}$ has an isolated vertex or has a pendant vertex whose deletion leaves a vertex-transitive graph.*

Proof. We have $\text{drn}(G) = 1$ if and only if the dadeck of G has a dacard (C, d) as described in Theorem 2.4. If C is complete or edgeless, or if $d \in \{0, |V(C)|\}$, then G or $\overline{G}$ has an isolated vertex. Case 2 of Theorem 2.4 yields the second possibility here. □

We close this section with a general bound for regular graphs. Regular graphs are well known to be reconstructible, since the degree list can be determined from the deck, and the deficient vertices in any card must be the neighbors of the missing vertex. One dacard gives the degree of the missing vertex, but it does not give the degree list and hence does not determine G. Nevertheless, we obtain an upper bound on $\text{drn}(G)$.

Theorem 2.6. *If G is a k-regular graph on n vertices, then* $\mathrm{drn}(G) \le \min\{k+2, n-k+1\}$.

Proof. Since G is k-regular, each card has k vertices of degree $k-1$ and $n-1-k$ vertices of degree k. Let H be a graph that shares $k+2$ dacards with G. Let (C,k) be one such dacard, with $C = G - v$, so there exists $u \in V(H)$ with $C = H - u$.

If $H \not\cong G$, then u has a neighbor w in H with degree k in C, and $\Delta(H) = k+1$. The $k+2$ given dacards of H imply that H has at least $k+2$ vertices of degree k whose deletion from H leaves a subgraph with maximum degree at most k. Since $d_H(w) = k+1$, deleting w cannot yield a dacard of G. Hence vertices in H whose deletion yields a dacard of G lie in $N_H(w)$. There are only $k+1$ such vertices, so any graph agreeing with G on $k+2$ dacards must be isomorphic to it, and $\mathrm{drn}(G) \le k+2$.

The complement of a k-regular graph is $(n-1-k)$-regular, so Lemma 2.3 and the argument above yield $\mathrm{drn}(G) = \mathrm{drn}(\overline{G}) \le (n-1-k)+2$, completing the proof. $\square$

Equality holds in the bound of Theorem 2.6 for graphs of the form $tK_{m,m}$ with $t > 1$, proved by Ramachandran [45]. Ramachandran [45] also proved for $k, t \ge 2$ that if G is a connected k-regular graph on n vertices, where $n \ge 3$, then $\mathrm{drn}(tG) \le n-k+2$.

2.3 Vertex-transitive graphs

For regular graphs that are vertex-transitive, we obtain sharper results. Observe that a graph is vertex-transitive if and only if its dacards are pairwise isomorphic. Since vertex-transitive graphs are regular, Theorem 2.6 provides an upper bound. We will prove further lower and upper bounds and give sufficient conditions for equality in the bounds.

Since $\text{drn}(G) = 2$ almost always, only special graphs need more dacards. When the dacards are identical, there is no clever choice of dacards, so it is natural to expect vertex-transitive graphs to be hard to reconstruct. Ramachandran [45] showed that $\text{drn}(tK_{m,m}) = m + 2$ when $t > 1$. On the other hand, the value can remain small: for $t, m > 1$, Ramachandran [45] showed that $\text{drn}(tK_m) = 3$ even though $\text{rn}(tK_m) = m + 2$ (Myrvold [40]). Note that by setting $t = 2$ and applying $\text{drn}(\overline{G}) = \text{drn}(G)$, one also obtains $\text{drn}(K_{m,m}) = 3$.

Definition 2.7. A *twin* of v is a vertex having the same neighborhood as v. A *clone* of a vertex x in a graph is a vertex having the same closed neighborhood as x.

Theorem 2.8. *If G is vertex-transitive but is not complete or edgeless, then $\text{drn}(G) \geq 3$.*

Proof. Let (C, d) denote the only dacard of G. To show that $\text{drn}(G) > 2$, we construct a graph H different from G that has at least two copies of (C, d) in its dadeck.

Let v be a vertex of G, so $C = G - v$. If every neighbor of v in G is a clone of v, then G is a disjoint union of complete graphs, say mK_r with $m \geq 2$ and $r \geq 2$. In this case, $C = (m-1)K_r + K_{r-1}$. Let x be a nonneighbor of v in G. Form H by adding to C a vertex x' with the same neighborhood as x. Now H has a noncomplete component, so $H \not\cong G$, but $H - x' = H - x = C$, so H has (C, d) as a dacard twice.

Otherwise, let u be a neighbor of v with $N_G[u] \neq N_G[v]$. There exist vertices $w \in N_G(u) - N_G(v)$ and $w' \in N_G(v) - N_G(u)$. Form H by adding to C a vertex u' such that $N_H[u'] = N_G[u] - \{v\}$. Note that $d_H(w) > d_C(w) > d_C(w') = d_H(w')$, so H is not regular; thus $H \not\cong G$. Since $N_H[u'] = N_H[u]$, we have $H - u = H - u' = C = G - v$, so H has (C, d) as a dacard twice. $\square$

We will shortly give sufficient conditions for equality in this bound. The bound can be arbitrarily bad, since Ramachandran proved that $\mathrm{drn}(tK_{m,m}) = m+2$. We next extend Ramachandran's result by proving this value for a more general family of vertex-transitive graphs. The graph produced is $tK_{m,m}$ when the construction begins with tK_2.

Definition 2.9. An *expansion* of a graph G is a graph H obtained by replacing each vertex of G with an independent set such that copies in H of two vertices of G are adjacent in H if and only if the original vertices were adjacent in G. The *m-fold expansion* $G^{(m)}$ is the expansion of G in which each vertex expands into an independent set of size m. A *twin-set* in a graph is a maximal vertex subset containing vertices with identical neighborhoods.

Theorem 2.10. *Let G be a vertex-transitive graph other than a complete graph, and suppose that G has no twins. If $m \geq 2$, then* $\mathrm{drn}(G^{(m)}) = m+2$.

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$. In $G^{(m)}$, each vertex v_i of G becomes an independent set V_i of size m. All vertices in V_i have the same neighborhood, while vertices in distinct such sets have different neighborhoods, since G has no twins. Thus the sets $V_1, \ldots, V_n$ are twin-sets. Note that $G^{(m)}$ is vertex-transitive and km-regular, where G is k-regular, and every vertex neighborhood in $G^{(m)}$ is a union of twin-sets. Let C be the unique card of $G^{(m)}$.

We first show that $\mathrm{drn}(G^{(m)}) \geq m+2$. Since G is not a complete graph and has no twins, it has nonadjacent vertices v_i and v_j with distinct neighborhoods. View C as $G - x$, where $x \in V_i$. Construct H by adding to C a vertex u with neighborhood $N(V_j)$ (the common neighborhood of all vertices of V_j). Since $x \notin N(V_j)$, we have $d_H(u) = km$. In $G^{(m)}$ every set of $m+1$ vertices contains two vertices having distinct neighborhoods, but in H the $m+1$ vertices in $V_j \cup \{u\}$ all have the same neighborhood. Hence $H \not\cong G^{(m)}$. Furthermore, the dacards for these vertices of H are the same as the dacards for $G^{(m)}$. Thus $\mathrm{drn}(G^{(m)}) \geq m+2$.

Now let H be a graph having vertices $u_1, \ldots, u_{m+2}$ of degree km such that $H - u_i \cong C$ for $1 \leq i \leq m+2$. Since $m \geq 2$, there are n twin-sets in C, one of which has size $m-1$; call it U. Treating a deleted vertex of $G^{(m)}$ (assume it is u_1) as a member of V_1, we may let $V_1 - \{u_1\}$, $V_2, \ldots, V_n$ be the twin-sets of C. There are exactly n distinct vertex neighborhoods in C. Suppose that $N_H(u_1)$ is none of these. Since $|U| = m-1$, among $u_2, \ldots, u_{m+2}$ there is a vertex u_j not in U. In $C - u_j$, there remain n distinct neighborhoods (since the n twin-sets of C remain nonempty), and none of them is $N_H(u_1) - \{u_j\}$. Replacing u_1, we find that $H - u_j$ has $n+1$ distinct neighborhoods, contradicting $H - u_j \cong C$.

Thus $N_H(u_1)$ is a vertex neighborhood in C. If it is the neighborhood of the deficient set, then $H \cong G^{(m)}$. Otherwise, H is an expansion of G in which one twin-set T has size $m+1$, one twin-set U has size $m-1$, and the others have size m. The only way to delete a vertex from H so that the twin-sets in the resulting graph have the same sizes as in C is to delete a vertex of T. Since $|T| = m+1$, the dacard (C, km) cannot occur $m+2$ times for H. □

In a vertex-transitive graph, the twin-sets all have the same size.

Corollary 2.11. *If G is a vertex-transitive graph other than a complete multipartite graph, then* $\mathrm{drn}(G^{(m)}) = tm + 2$ *for every* $m \geq 2$*, where* t *is the size of the twin-sets in* G*.*

Proof. Collapsing the twin-sets of G into single vertices yields a vertex-transitive graph G_0 having no twins, and $G = G_0^{(t)}$. Since G is not a complete multipartite graph, G_0 is not a complete graph. Hence Theorem 2.10 applies to G_0, and $\mathrm{drn}(G^{(m)}) = \mathrm{drn}(G_0^{(tm)}) = tm + 2$. □

In the remainder of this section we study sharpness in the lower bound of

Theorem 2.8. We give a sufficient condition for $\mathrm{drn}(G) = 3$ in the family of vertex-transitive graphs and show that hypercubes and some other products satisfy it.

Definition 2.12. A vertex-transitive graph G is *coherent* if a card C of G formed by adding one vertex z to a two-vertex-deleted subgraph $G - \{x, y\}$ can only be formed by making z adjacent to $N_{G-y}(x)$ or $N_{G-x}(y)$.

Coherence prevents the deletion of two vertices from G in such a way that the card can be recreated by adding a vertex adjacent to some set of deficient vertices other than the full neighborhood of one of the deleted vertices.

Theorem 2.13. *Let G be a k-regular vertex-transitive graph. If G is coherent and has no clones or twins, then* $\mathrm{drn}(G) = 3$.

Proof. Let C be the unique card of G. We must show that if some graph H has vertices u, v, w of degree k such that deleting any one yields C, then $H \cong G$.

Let S be the set of vertices of degree $k-1$ in $H-u$. Since $H-u \cong C \cong G-x$, we may assume that $H-u = G-x$ (using the same vertex names), so $N_G(x) = S$ and $|S| = k$. Now $H - u - v$ is obtained by deleting x and v from G. The card $H - v$ is obtained by adding u and the appropriate edges to $H - u - v$; doing this adds u and appropriate edges to $G - x - v$ to produce a graph isomorphic to C. By coherence, $N_{H-v}(u)$ is $N_{G-v}(x)$ or $N_{G-x}(v)$.

If $N_{H-v}(u) = N_{G-v}(x)$, then $S - \{v\} \subseteq N_H(u)$. Also, $|S - \{v\}|$ is $k - 1$ or k, depending on whether $v \in N_G(x)$. Since we are given $d_H(u) = k$, we obtain $N_H(u) = S$ and $H \cong G$.

If $N_{H-v}(u) = N_{G-x}(v)$, then $|N_H(u) \cap N_H(v)| \in \{k - 1, k\}$, depending on whether $v \in N_G(x)$. This makes u and v clones or twins in H, respectively, since $d_H(u) = k$. Now we look at $H - w$. Whether w is adjacent to neither or both of $\{u, v\}$ in H, still u and v are clones or twins in $H - w$. Since G is regular, $H - w \cong C \cong G - x$, and $d_{H-w}(u) = d_{H-w}(v)$, forming G from $H - w$ makes w

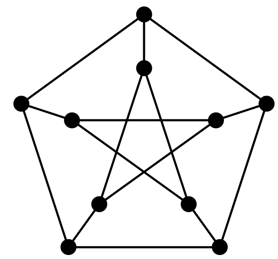


Figure 2.1: The Petersen graph.

adjacent to neither or both of $\{u, v\}$. As a result, u and v are clones or twins in G, which contradicts the prohibition of such pairs. □

It is easy to see that $tK_{m,m}$ and tK_m are coherent, but $tK_{m,m}$ has twins and tK_m has clones. We have noted that $\text{drn}(tK_{m,m}) = m + 2$ and $\text{drn}(tK_m) = 3$.

Proposition 2.14. *If G is a coherent 2-connected vertex-transitive graph, then tG is coherent.*

Proof. Vertices u and v to be deleted from tG may lie in the same component or not. If they don't, then a vertex added to turn $tG - u - v$ into the card C must restore one of the components of G. If u and v lie in the same component of tG, then the needed property follows from the coherence of G. □

We close this section with several natural examples to illustrate the role of coherence.

The *Petersen graph* is shown in Figure 2.1; it is the graph whose vertices are the 2-element subsets of a set of five elements, with two vertices adjacent if and only if the associated subsets are disjoint.

Example 2.15. If G is the Petersen graph, then $\text{drn}(G) = 3$. Any two nonadjacent vertices in G have exactly one common neighbor, and any two adjacent vertices have no common neighbors; hence G has no twins or clones. It therefore suffices to check coherence. Let C be the card. There are only two types of vertex pairs in G; adjacent or nonadjacent.

Deleting two adjacent vertices leaves four vertices with degree 2. Any two of them that did not have a common neighbor among the deleted vertices have a common neighbor among the remaining vertices. Adding a vertex adjacent to both of them creates a 4-cycle, which does not exist in C.

Deleting two nonadjacent vertices leaves one vertex with degree 1, and the vertices having degree 2 induce $2K_2$. A vertex added to form C must be adjacent to the leaf and to one vertex from each edge of this $2K_2$. To avoid creating a 4-cycle, only two of the four such choices are allowable, and these yield the vertex neighborhoods of the deleted vertices.

We next consider the k-dimensional hypercube Q_k, the graph with vertex set $\{0,1\}^k$ in which two vertices are adjacent if and only if they differ in exactly one coordinate.

It is well known that vertices separated by distance 2 in Q_k have exactly two common neighbors.

Theorem 2.16. *If* $k \geq 2$, *then* $\mathrm{drn}(Q_k) = 3$.

Proof. The lower bound follows from Theorem 2.8. Ramachandran [45] showed that $\mathrm{drn}(C_4) = 3$. Since $Q_2 \cong C_4$, we may assume that $k \geq 3$. Since Q_k has no clones or twins, it suffices by Theorem 2.13 to show that Q_k is coherent. Let C be the unique card of Q_k. Given $u, v \in V(Q_k)$, let $F = Q_k - \{u, v\}$, and let $S = N_{Q_k - v}(u)$ and $S' = N_{Q_k - u}(v)$. Let z be a vertex added to F to obtain C; we must show that $N_C(z) \in \{S, S'\}$.

The vertex z cannot have have neighbors in both partite sets of F, since C is bipartite. Also it has no neighbor with degree k in F, since $\Delta(C) \leq k$. Hence $N_C(z) \in \{S, S'\}$ when u and v lie in opposite partite sets.

Now consider u and v in the same partite set. Since $\delta(C) = k-1$ and $\Delta(C) \le k$, we have $S \cap S' \subseteq N_C(z) \subseteq S \cup S'$. If $N_C(z) \notin \{S, S'\}$, then z has neighbors in both $S - S'$ and $S' - S$. Since $d_C(z) = k = |S| = |S'|$, there also exist $w \in S - S'$ and $w' \in S' - S$ outside $N_C(z)$. Now $d_C(w) = d_C(w') = k-1$. Hence w and w' have a common neighbor in Q_k deleted to obtain C. Since the distance between them in Q_k is 2, they have exactly two common neighbors in Q_k, and hence exactly one remains in C. However, since by choice neither lies in $S \cap S'$, neither u nor v is one of their common neighbors. Hence their common neighbors in Q_k both remain in F and hence in C. The contradiction implies that $N_C(z) \in \{S, S'\}$. $\square$

The hypercube Q_k is the cartesian product of k factors isomorphic to K_2. It would be nice to generalize Theorem 2.16 to all cartesian products of complete graphs. Our next result does this for one special case. As noted in Chapter 1, the cartesian product $G \square K_2$ is also called the *prism over G*.

Unfortunately, $K_3 \square K_2$ is not coherent, since it has C_4 as a double-vertex-deleted subgraph, and the card can be obtained by adding z adjacent to any two consecutive vertices on the cycle. Hence we cannot apply Theorem 2.13 to this graph.

Lemma 2.17. $\mathrm{drn}(K_3 \square K_2) \le 3$.

Proof. It suffices to show that three dacards determine $K_3 \square K_2$. Let C be the unique card of $K_3 \square K_2$, and consider a graph H having three cards isomorphic to C, obtained by deleting any one of $\{u, v, w\}$, all having degree 3 in H.

If H has a vertex x of degree 4, then $\{u, v, w\} \subseteq N_H(x)$, since $\Delta(C) = 3$. Let z be a vertex added to C to form H. Since x has only one neighbor with degree 3 in C, z is adjacent to a neighbor of x with degree 2 in C. If z is also adjacent to the unique nonneighbor y of x, then z is a clone or twin of a vertex t in H and hence in one of $\{H-u, H-v, H-w\}$. Since C has no clones or twins, this is a contradiction. Thus $d_H(y) \le 2$. Since $xy \notin E(H)$, deleting one of $\{u, v, w\}$ leaves $d_C(y) \le 1$. This contradicts $\delta(C) = 2$.

Hence $\Delta(H) = 3$, which yields $G \cong K_3 \square K_2$. $\square$

Theorem 2.18. *If $k \ge 2$, then* $\mathrm{drn}(K_k \square K_2) = 3$.

Proof. Again the lower bound is from Theorem 2.8. Let $G = K_k \square K_2$. We have observed that $\mathrm{drn}(G) \le 3$ when $k \le 3$, so consider $k \ge 4$. Let C be the unique card of G. Since G has no clones or twins, by Theorem 2.13 it suffices to show that G is coherent. Given $u, v \in V(G)$, let $F = G - \{u, v\}$, and let $S = N_{G-v}(u)$ and $S' = N_{G-u}(v)$. Let z be a vertex added to F to obtain C; we must show that $N_C(z) \in \{S, S'\}$. Let A and B be the two k-cliques in G. By symmetry, we have two cases.

Case 1: $u, v \in A$. Vertices remaining in A have degree $k-2$ in F, and the neighbors of u and v in B have degree $k-1$ in F. Since $\delta(C) = k-1$ and $\Delta(C) = k$, we conclude that $N_C(z)$ contains all of $A - \{u, v\}$ and the neighbor of u or v in B. Hence $N_C(z) \in \{S, S'\}$.

Case 2: $u \in A$, $v \in B$. Here $F \subseteq K_{k-1} \square K_2$, with equality if $uv \in E(G)$ and one missing "cross-edge" if $uv \notin E(G)$. Since $k \ge 4$, the only $(k-1)$-cliques in F are $A - u$ and $B - v$. Since C has a k-clique, z must be adjacent to all of $A - u$ or $B - v$. Since C has exactly k vertices of degree $k-1$, z has no other neighbor if $uv \in E(G)$ and is adjacent to the remaining vertex of degree $k-2$ in F if $uv \notin E(G)$. In either case, $N_C(z) \in \{S, S'\}$. $\square$

Similar arguments can be made for other families of vertex-transitive graphs. For example, it follows also that $\mathrm{drn}(C_k \square K_2) = 3$ for $k \ge 3$, where C_k is the k-cycle. We ask which vertex-transitive graphs are coherent, or at least which vertex-transitive graphs have coherent cartesian products with K_2.

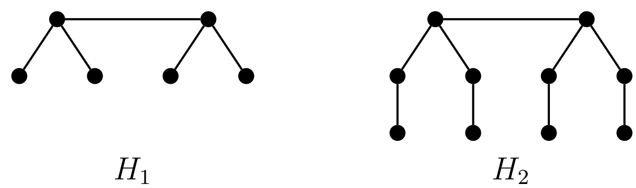


Figure 2.2: Trees with degree-associated reconstruction number 3.

2.4 Trees

In one of the first papers on reconstruction, Kelly [30] proved that trees with at least three vertices are reconstructible. Several papers have studied reconstruction of trees given only some of the cards from the deck. Harary and Palmer [21] showed that every tree is uniquely determined by its leaf-deleted subgraphs, and Lauri [32] showed that every tree with at least three cut-vertices is reconstructible from its cut-vertex-deleted subgraphs.

Myrvold [41] proved that every tree with at least 5 vertices has reconstruction number 3. Together with Corollary 2.5, this implies the following.

Corollary 2.19. *If T is a tree, then* $\mathrm{drn}(T) \le 3$*, and* $\mathrm{drn}(T) = 1$ *if and only if T is a star.*

By Corollary 2.2, almost every graph has degree-associated reconstruction number 2, and Prince [43] proved the "almost-always" statement also for the class of all trees. The trees H_1 and H_2 in Figure 2.2 do satisfy $\mathrm{drn}(H_1) = \mathrm{drn}(H_2) = 3$.

Example 2.20. $\mathrm{drn}(H_1) = 3$. The graph H_1 has only two distinct dacards. They are $(P_3 + 2K_1, 3)$ and $(S, 1)$, where S is the tree obtained by subdividing one edge of $K_{1,3}$ (that is, replacing an edge uv by a vertex w and two edges uw and wv). Hence there are three ways to take two dacards; two of the first, two of the second, and one of each. For these three cases, other graphs having the same two dacards are the graph obtained from $2K_1 + K_4$ by deleting one edge, the tree obtained from

$K_{1,4}$ by subdividing one edge, and the tree obtained from $K_{1,3}$ by subdividing one edge twice, respectively.

Three dacards suffice, using one leaf and the two central vertices. For any reconstruction G, the leaf card forces G to be a tree, and the other two force G to have two vertices of degree 3. Hence G is obtained from S by appending a leaf to the one vertex of degree 2.

The argument for H_2 is similar but longer. We have particular interest in H_1 because it lies in the family we will study for the rest of this chapter. First, the fact that we know of no tree T other than H_1 and H_2 such that $\mathrm{drn}(T) = 3$ suggests a conjecture.

Conjecture. *Only finitely many trees T satisfy* $\mathrm{drn}(T) = 3$.

A *caterpillar* is a tree whose non-leaf vertices induce a path called the *spine* of the caterpillar. In the remainder of this chapter, we prove that the tree H_1 and the stars $K_{1,m}$ are the only caterpillars T such that $\mathrm{drn}(T) \neq 2$.

By Corollary 2.19, it suffices to prove that $\mathrm{drn}(T) \leq 2$ for caterpillars other than H_1. In this section we give sufficient conditions for $\mathrm{drn}(T) \leq 2$ when T is a tree. In the subsequent sections of this chapter, we prove this inequality for various classes of caterpillars described by conditions on the list of degrees of the spine vertices, culminating in the full proof. The task is to select for each caterpillar T a pair of dacards that together determine T.

The *skeleton* of a tree T is the subtree T' obtained by deleting all leaves from T. Thus caterpillars are the trees whose skeletons are paths, and the spine of a caterpillar is its skeleton. We use $C(a_1, \ldots, a_s)$ to denote a caterpillar with spine $\langle v_1, \ldots, v_s \rangle$ by attaching a_i leaf neighbors to v_i for each $i \in \{1, \ldots, s\}$. We call $(a_1, \ldots, a_s)$ the *spine list*. Note that $C(a_1, \ldots, a_s) \cong C(a_s, \ldots, a_1)$ and that a_1 and a_s are both positive. Where convenient, we denote a repeated string in this notation by enclosing it in parentheses and writing its multiplicity as a superscript in parentheses. For example, $C(a, b, c, d, b, c, d, b, c, d, e, f) = C(a, (b, c, d)^{(3)}, e, f)$.

The *weight* $w(u)$ of a vertex u in a tree T is the maximum number of vertices in a component of $T-u$; note that all leaves in an n-vertex tree have weight $n-1$. The *centroid* of a tree is the set of vertices having minimum weight. Myrvold [41] used centroids of trees extensively in her analysis of reconstruction number of trees. To keep our presentation self-contained, we include short proofs of some elementary observations.

Lemma 2.21 (Myrvold [41])**.** *The centroid of an n-vertex tree T consists of one vertex or two adjacent vertices. Also, $w(v) \leq n/2$ if and only if v is in the centroid of T, and the centroid of T has size* 1 *if and only if T has a vertex with weight strictly less than $n/2$.*

Proof. For each vertex v, mark an incident edge from v toward a largest component of $T - v$. Since T has n vertices and $n - 1$ edges, some edge ab is marked twice. Let A and B the vertex sets of the components of $T - ab$, with $a \in A$ and $b \in B$. Note that $w(a) = |B|$ and $w(b) = |A|$, so $w(a) + w(b) = n$.

If $w(a) = w(b) = n/2$, then $|A| = |B| = n/2$; for $c \in V(T) - \{a, b\}$, we have $w(c) > |B|$ if $c \in A$ and $w(c) > |A|$ if $c \in B$. Thus the centroid of T is $\{a, b\}$, the set of two adjacent vertices with weight at most $n/2$.

Suppose that $w(a) < n/2$. Let $C_1, \ldots, C_{d(a)}$ denote the vertex sets of the components of $T - a$. For a vertex $c \in C_i$, note that $T - c$ has a component of order at least $n - |C_i|$; hence $w(c) \geq n - |C_i| > n/2$, since $|C_i| \leq w(a) < n/2$. Thus the centroid of T is $\{a\}$ and consists of the single vertex with weight strictly less than $n/2$. A similar conclusion holds if $w(b) < n/2$. $\square$

A tree is *unicentroidal* or *bicentroidal* depending on whether its centroid has size 1 or 2, respectively. For simplicity, we refer to the centroid vertex of a unicentroidal tree as the centroid. A *centroidal vertex* is a vertex in the centroid.

Lemma 2.22 (Myrvold [41]). *Let v be the centroid in a unicentroidal tree T. If ℓ is a leaf in T, then v is centroidal in $T - \ell$.*

Proof. Let T have n vertices. By Lemma 2.21, $w(v) < n/2$. The weight of v in T' is at most $(n-1)/2$, since deleting ℓ simply reduces one component of $T - v$. By Lemma 2.21, v is centroidal in T'. $\square$

These facts about centroids can be useful in reconstructing a tree from its dacards. Note that if G has a card that is a tree obtained by deleting a vertex of degree 1, then G is a tree.

Proposition 2.23. *If T is a unicentroidal tree with a leaf ℓ adjacent to the centroid vertex, and $T - \ell$ is unicentroidal, then* $\mathrm{drn}(T) \leq 2$.

Proof. Let $T' = T - \ell$, and let $\hat{T}$ be the card obtained by deleting the centroid from T. Thus $(T', 1)$ and $(\hat{T}, d)$ are the corresponding dacards, and ℓ is an isolated vertex in $\hat{T}$.

Let G be a graph having these dacards, obtained by deleting vertices u and v, respectively. From the first dacard, G is a tree. From the sizes of the components of $\hat{T}$, Lemma 2.21 tells us that G is unicentroidal with centroid v.

Since u is a leaf and $G - u$ is unicentroidal (being isomorphic to T'), Lemma 2.22 identifies v in $G - u$ as the centroid of $G - u$, Since $\hat{T} = G - v$, the d components of $\hat{T}$ agree with the components obtained by deleting the centroid from T', except that one may have u as an extra leaf. However, we know from T that instead $\hat{T}$ has one more component than $T' - v$, an isolated vertex. This forces u to be adjacent to v in G, yielding $G \cong T$. $\square$

We have noted that having a dacard $(G - v, 1)$ in which $G - v$ is a tree forces G to be a tree. Our next lemma gives another sufficient condition on dacards for G to be a tree.

Lemma 2.24. *Let G be a graph with dacards $(A, 2)$ and $(B, 2)$. If A and B are forests with two components, and the sizes of the components of A do not equal those of B, then G is a tree.*

Proof. Let the sizes of the components in A and B be $\{a_1, a_2\}$ and $\{b_1, b_2\}$, respectively. Let u and v be the vertices such that $G - u = A$ and $G - v = B$.

If G is disconnected, then the neighbors of u in G belong to the same component of A, which we may call A_1. Now G has two components with orders $a_1 + 1$ and a_2, and the component of G containing A_1 is not a tree. To make B a forest, v must lie on all cycles in G and hence must lie in A_1. Since G and B both have two components, v is not a cut-vertex of A_1. Now $\{a_1, a_2\} = \{b_1, b_2\}$, a contradiction.

Hence G is connected. Since $d_G(u) = 2$, it follows that G is a tree. $\square$

By the characterization in Corollary 2.5, the only trees T for which $\mathrm{drn}(T) = 1$ are stars. We have also observed that $\mathrm{drn}(H_1) = 3$. To complete our analysis of caterpillars, in the remainder of this chapter we only need to prove results showing that caterpillars other than H_1 have degree-associated reconstruction number at most 2. General arguments for reconstruction of trees often must exclude the special case of paths; we treat them separately here.

Proposition 2.25. *If $n \geq 4$, then* $\mathrm{drn}(P_n) = 2$.

Proof. For $n = 4$, use the two dacards $(P_3, 1)$ and $(P_1 + P_2, 2)$. The first forces every reconstruction to be a tree, and hence in the second the missing vertex has a neighbor in each component, yielding P_4.

For $n \geq 5$, let $a = \lfloor \frac{n-1}{2} \rfloor$ and $b = \lceil \frac{n-1}{2} \rceil$. Let G be a graph having the two dacards $(P_a + P_b, 2)$ and $(P_{a-1} + P_{b+1}, 2)$, associated with u and v, respectively. By Lemma 2.24, G is a tree. (Here $a - 1 \geq 1$ requires $n \geq 5$.)

Let w be a neighbor of u in G. If w is not a leaf in $G - u$, then $d_G(w) = 3$. Since $\Delta(G - v) = 2$, we have $v \in N_G(w)$. Now the component of $G - v$ containing

u has at least $a+3$ vertices, since it contains all of one component of P_a+P_b plus u, w, and another neighbor of w. Since the components of $G-v$ have at most $a+2$ vertices, we conclude that u has no neighbor with degree 3 in G, and hence $G = P_n$. □

Our general arguments fail also for several other classes of caterpillars where we will need alternative choices of dacards. It is worth noting that P_n is forced by two dacards only when they correspond to a centroidal vertex and a noncentroidal neighbor of the centroid.

2.5 Caterpillars of the form $C(1, 0, a_3, \ldots, a_{s-2}, 0, 1)$

We begin with a technical lemma that will restrict the form of caterpillars with special symmetry properties. A *palindrome* is a list unchanged under reversal.

Lemma 2.26. *Let $B = (b_1, \ldots, b_s)$. If $(b_1, \ldots, b_s)$ and $(b_3, \ldots, b_s)$ are palindromes, then either B is constant, or s is odd and B alternates two values. If $(b_1, \ldots, b_{s-1})$ and $(b_2, \ldots, b_s)$ are palindromes, then either B is constant, or s is even and B alternates two values.*

Proof. Define a graph R with vertex set $\{v_1, \ldots, v_s\}$ such that $v_iv_j \in E(R)$ if and only if the palindrome requirements force $b_i = b_j$. If R consists of one component, then B is constant. If R consists of two components, one containing the odd-indexed and the other the even-indexed vertices, then B is constant or alternates between two values.

If $(b_1, \ldots, b_s)$ and $(b_3, \ldots, b_s)$ are palindromes, then $v_iv_j \in E(R)$ if and only if $i + j \in \{s + 1, s + 3\}$. If s is even, then R is the path $\langle v_1, v_s, v_3, v_{s-2}, \ldots, v_{s-1}, v_2\rangle$. If s is odd, then R consists of two paths $\langle v_1, v_s, v_3, v_{s-2}, \ldots\rangle$, containing the odd-indexed vertices, and $\langle v_2, v_{s-1}, v_4, v_{s-3}, \ldots\rangle$, containing the even-indexed vertices.

If $(b_1, \ldots, b_{s-1})$ and $(b_2, \ldots, b_s)$ are palindromes, then $v_iv_j \in E(R)$ if and only if $i+j \in \{s, s+2\}$. If s is odd, then R is the path $\langle v_1, v_{s-1}, v_3, v_{s-3}, \ldots, v_{s-2}, v_2, v_s\rangle$. If s is even, then R consists of two paths $\langle v_1, v_{s-1}, v_3, v_{s-3}, \ldots\rangle$, containing the odd-indexed vertices, and $\langle v_s, v_2, v_{s-2}, v_4 \ldots\rangle$, containing the even-indexed vertices. $\square$

In the remainder of this chapter, $T = C(1, 0, a_3, \ldots, a_{s-2}, 0, 1)$, with spine $\langle v_1, \ldots, v_s\rangle$ such that v_i has a_i leaf neighbors for $1 \le i \le s$. By Proposition 2.25, $\mathrm{drn}(P_{s+2}) = 2$. Since P_{s+2} is the case $a_3 = \cdots = a_{s-2} = 0$, we may let $r = \min\{i\colon\ a_i > 0 \text{ and } 3 \le i \le s-2\}$. To show $\mathrm{drn}(T) \le 2$, we present two dacards that determine T. Consider the dacards for leaves adjacent to v_1 and v_r, writing

$$C_1 = C(1, 0^{(r-3)}, a_r, \ldots, a_{s-2}, 0, 1), \qquad D_1 = (C_1, 1),$$
$$C_2 = C(1, 0^{(r-2)}, a_r - 1, a_{r+1}, \ldots, a_{s-2}, 0, 1), \qquad D_2 = (C_2, 1).$$

Let G be a graph reconstructed from dacards D_1 and D_2, with vertices u and v being the deleted vertices, respectively. Since $d_G(u) = d_G(v) = 1$, either card forces G to be a tree. We show that $G \cong T$, with some exceptions where we will later use other dacards.

Lemma 2.27. *If $T = C(1, 0, a_3, \ldots, a_{s-2}, 0, 1)$ and T is not a path, then the dacards D_1 and D_2 determine T in all cases except when T satisfies one of the following conditions:*

(1) *$T = C(1, 0^{(p)}, 1, 0^{(q)}, 1)$, where $p, q \ge 1$;*

(2) *$T = C(1, 0^{(p+1)}, k, (\alpha), k-1, 0^{(p)}, 1)$, where $k \ge 1$, $p \ge 0$, and (α) is a palindrome.*

Proof. From D_2 it follows that G is a tree with diameter at least $s+1$. Since $\mathrm{diam}(G-u) = s$ and $s \ge 5$, it follows that u is adjacent in G to an endpoint of a longest path in $G-u$. Hence G is T or is $C(1, 0^{(r-3)}, a_r, \ldots, a_{s-2}, 0, 0, 1)$. Suppose the latter.

Since $G - v \cong C_2$, and both G and C_2 have spines with s vertices, decreasing one term of the spine list L for G yields the spine list L' for C_2 or its reverse, L''. Let L_i, L'_i, L''_i denote the ith entry in L, L', L'', respectively. Since $L_{r-1} = a_r > 0 = L'_{r-1}$, changing L into L' by decreasing one L_i requires $i = r-1$ and $a_r = 1$. Since no other change is allowed, we have

$$\begin{array}{ccccccccl}
a_r - 1 & = & L_{r-1} - 1 & = & L'_{r-1} & = & 0, & & \\
a_{r+1} & = & L_r & = & L'_r & = & a_r - 1, & & \\
a_{i+1} & = & L_i & = & L'_i & = & a_i & & \text{for } r+1 \le i \le s-3, \\
0 & = & L_{s-2} & = & L'_{s-2} & = & a_{s-2}, & &
\end{array}$$

and thus $a_r = 1$ and $a_{r+1} = \cdots = a_{s-2} = 0$. Hence $T = C(1, 0^{(r-2)}, 1, 0^{(s-r-1)}, 1)$, as in (1).

Suppose instead that decreasing some L_j by 1 changes L into L''; we first restrict the choices for j. By construction, $3 \le r \le s-2$ and $s \ge 5$. We compare the expressions below.

$$\begin{array}{rllllll}
T = C(1, & 0^{(r-2)}, & a_r, & \ldots, & a_{s-2}, & 0, & 1) \\
G = C(1, & 0^{(r-3)}, a_r, & & \ldots, & a_{s-2}, \;\; 0, & 0, & 1) = C(L) \\
C_2 = C(1, 0, a_{s-2}, \ldots, & a_{r+1}, & a_r - 1, & & 0^{(r-2)} & , & 1) = C(L'') \\
\text{positions} = \quad 1, 2, & 3, \; \ldots, s-r, & s-r+1, & \ldots, & s-2, & s-1, & s
\end{array}$$

Since $L_i = a_{i+1}$ for $2 \le i \le s-2$, we have $L_{r-1} + L_{s-r+1} = a_r + a_{s-r+2}$. Since $L''_i = a_{s+1-i}$ for $i \ne s-r+1$ (and $L''_{s-r+1} = a_r - 1$), setting $i = r-1$ yields $L''_{r-1} + L''_{s-r+1} = a_{s-r+2} + a_r - 1$, except that $L''_{r-1} + L''_{s-r+1} = a_{s-r+2} + a_r - 2$ when $r-1 = s-r+1$. In either case, $L''_{r-1} + L''_{s-r+1} < L_{r-1} + L_{s-r+1}$, and hence $j \in \{r-1, s-r+1\}$.

Since $L_i = 0$ for $2 \le i \le r-2$, we have $j \ge r-1$. Since only position j changes, the first $r-2$ positions agree in L and L''. Hence $a_i = 0$ for $s-r+3 \le i \le s-1$ (when $r=3$ this conclusion is empty). If $r-1 \ge s-r+2$, then this statement includes $a_r - 1 = 0$, since $L''_{s-r+1} = a_r - 1$. In this case $T = C(1, 0^{(r-2)}, 1, 0^{(s-1-r)}, 1)$, which satisfies description (1). If $r-1 = s-r+1$, then $s-r+3 = r+1$; we obtain $T = C(1, 0^{(r-2)}, a_r, 0^{(r-3)}, 1)$ and $G = C(1, 0^{(r-3)}, a_r, 0^{(r-2)}, 1)$, and hence $G \cong T$.

Hence we may assume that $r-1 < s-r+1$. Now $a_{i+1} = L_i = L''_i = a_{s+1-i}$ for $r \le i \le s-r$. Hence $(a_{r+1}, \dots, a_{s-r+1})$ is a palindrome, and a_{s-r+2} equals $a_r - 1$ (if $j = r-1$) or a_r (if $j = s-r+1$). Letting $\alpha = (a_{r+1}, \dots, a_{s-r+1})$, we have $T = C(1, 0^{(r-2)}, k, (\alpha), k', 0^{(r-3)}, 1)$ and $G = C(1, 0^{(r-3)}, k, (\alpha), k', 0^{(r-2)}, 1)$, where $k = a_r \ge 1$ and $k' \in \{k, k-1\}$. If $k' = k$, then $G \cong T$; otherwise, T satisfies description (2).

□

Since $C(a_1, \dots, a_s) \cong C(a_s, \dots, a_1)$ for every caterpillar by reversing the spine, we have shown that a caterpillar of the form $C(1, 0, a_3, \dots, a_{s-2}, 0, 1)$ is determined by the stated choice of dacards taken from one end or the other unless under both directions the caterpillar has one of the exceptional forms in described in Lemma 2.27.

Our argument to handle these exceptional forms has exceptions itself. The difficulty is that in the exceptional cases the two dacards D_1 and D_2 chosen for Lemma 2.27 do not determine T. Nevertheless, in all exceptional cases, we find two dacards that work. We show first that the type (1) exceptional form in Lemma 2.27 causes no difficulty.

Proposition 2.28. *If $T = C(1, 0^{(p)}, 1, 0^{(q)}, 1)$, where $p, q \ge 0$, then* $\mathrm{drn}(T) \le 2$.

Proof. The caterpillar T contains one vertex of degree 3, which has exactly one leaf neighbor. Use the dacards for these two vertices: $D_1 = (P_{p+q+5}, 1)$ and $D_2 = (P_{p+2} + K_1 + P_{q+2}, 3)$. Let G be a reconstruction from these dacards, with u and v being the respective deleted vertices. As a leaf deletion, D_1 forces G to be a tree. Since $G - u$ is a path, v is the only vertex of degree 3 in G. Hence v must have a neighbor in each component of $P_{p+2} + K_1 + P_{q+2}$, and that neighbor cannot have degree 2 in its component. We obtain $G \cong T$. $\square$

Among the type (2) exceptions in Lemma 2.27, we consider several special forms.

Proposition 2.29. *If $T = C(1, 0^{(p+1)}, (2, 0)^{(q)}, 1, 0^{(p)}, 1)$, where $p, q \geq 1$, then $\mathrm{drn}(T) \leq 2$.*

Proof. Let $j = p + 3 + 2\lfloor q/2 \rfloor$. The spine vertex v_j has degree 4. Consider the dacards obtained by deleting v_j or a leaf ℓ adjacent to v_j. Deleting ℓ leaves a tree with $2p+4q+6$ vertices, and hence any reconstruction G is a tree with $2p+4q+7$ vertices. The card when we delete v_j consists of two isolated vertices and two caterpillars, which have $p + 3 + 4\lfloor q/2 \rfloor$ and $p + 1 + 4\lceil q/2 \rceil$ vertices. For either parity of q, the maximum of these is $p + 3 + 2q$.

Let u and v be the leaf and the non-leaf vertices deleted from G to obtain these dacards. Since $p + 3 + 2q < (2p + 4q + 7)/2$, Lemma 2.21 implies that v is the centroid of G. The tree $G - u$ has $2p + 4q + 6$ vertices and is bicentroidal, with centroid vertices v_j and $v_{j\pm1}$ ($+1$ when q is odd, -1 when q is even); each of these vertices has weight $p + 2q + 3$. By Lemma 2.22, v is one of these two vertices. Since $d_G(v) = 4$ and the spine neighbors of v_j have no leaf neighbors, $v = v_j$. Since $d_{G-u}(v_j) = 3$, we obtain G from the leaf card $G - u$ by adding u adjacent to v_j. Thus $G \cong T$. $\square$

Proposition 2.30. *If $T = C(1, 0^{(p)}, 1^{(q)}, 0^{(p)}, 1)$, where $p \geq 1$ and $q \geq 0$, then $\mathrm{drn}(T) \leq 2$.*

Proof. If $q = 0$, then T is a path, and Proposition 2.25 applies. If $q = 1$, then Proposition 2.28 applies. Now consider $q \geq 2$. Note that $s = 2p + q + 2$, so $\mathrm{diam}(T) = 2p + q + 3$.

Let x be the leaf adjacent to v_{p+2}. Consider the dacards obtained by deleting v_p (with degree 2) and x. Note that $T - x = C(1, 0^{(p+1)}, 1^{(q-1)}, 0^{(p)}, 1)$ and $T - v_p = P_p + C(2, 1^{(q-1)}, 0^{(p)}, 1)$. Let G be a reconstruction from these two dacards, with $G - u \cong T - x$ and $G - v \cong T - v_p$. As usual, the leaf dacard forces G to be a tree. Since $\mathrm{diam}(G - u) = 2p + q + 3 = \mathrm{diam}\, T$, the neighbors of v in G must be endpoints of longest paths in the two components of $G - v$. Hence $G \cong T$ or $G = C(2, 1^{(q-1)}, 0^{(2p+1)}, 1)$, depending on which end of the longest path in the non-path component in $G - v$ is adjacent to v.

In the latter case, since the spine endpoints in $G - u$ each have only one leaf neighbor, u must be adjacent in G to the spine vertex having two leaf neighbors. Now $G - u \cong C(1^{(q)}, 0^{(2p+1)}, 1)$. Since $p \geq 1$ and $q \geq 2$, this graph is not isomorphic to $T - x$, a contradiction. Hence this case does not arise, and $G \cong T$. □

We now have the tools to prove the main result of this section.

Theorem 2.31. *If* $T = C(1, 0, a_3, \ldots, a_{s-2}, 0, 1)$, *then* $\mathrm{drn}(T) = 2$.

Proof. By Proposition 2.25, we may assume that T is not a path. In Lemma 2.27, we proved that the dacards for the leaves adjacent to v_1 and the next spine vertex having a leaf neighbor determine T unless both T and its reverse description $C(a_s, \ldots, a_1)$ have the forms specified in Lemma 2.27. If the description is as in (1) of Lemma 2.27, then T is a path plus one pendant edge, and Proposition 2.28 yields $\mathrm{drn}(T) \leq 2$.

Hence we may assume that both T and the reverse description T' are as in (2) of Lemma 2.27. If $L = (a_1, \ldots, a_s)$, then

$$L = (1, 0^{(p+1)}, k, (\alpha), k-1, 0^{(p)}, 1) = (1, 0^{(q)}, \ell - 1, (\beta), \ell, 0^{(q+1)}, 1)$$

for some palindromes (α) and (β) and integers p, q, k, ℓ such that $p, q \geq 0$ and $k, \ell \geq 1$.

Suppose that $k \geq 2$. The last nonzero entry of L before a_s is both a_{s-p-1} and a_{s-q-2}, so $q = p-1$ and $\ell = k-1$. Hence

$$L = (1, 0^{(p+1)}, k, (\alpha), k-1, 0^{(p)}, 1) = (1, 0^{(p-1)}, k-2, (\beta), k-1, 0^{(p)}, 1),$$

which implies that $k = 2$ and that both $(a_{p+4}, \ldots, a_{s-p-2})$ and $(a_{p+2}, \ldots, a_{s-p-2})$ are palindromes. Since $a_{p+2} = 0 \neq k = a_{p+3}$, Lemma 2.26 yields

$$T = C(1, 0^{(p+1)}, (2,0)^{(s/2-p-2)}, 1, 0^{(p)}, 1),$$

where s is even and $p \geq 1$. Since L contains at least one 2, Proposition 2.29 yields $\operatorname{drn}(T) \leq 2$.

By reversing L, the same argument holds when $\ell \geq 2$. Finally, when $k = \ell = 1$,

$$L = (1, 0^{(p+1)}, 1, (\alpha), 0^{(p+1)}, 1) = (1, 0^{(q+1)}, (\beta), 1, 0^{(q+1)}, 1).$$

Since $a_{p+3} = 1$ and $a_2 = \cdots = a_{q+2} = 0$, we have $p \geq q$. Since $a_{s-q-2} = 1$ and $a_{s-p-1} = \cdots = a_{s-1} = 0$, we have $q \geq p$. Thus $p = q$, and $(a_{p+4}, \ldots, a_{s-p-2})$ and $(a_{p+3}, \ldots, a_{s-p-3})$ are palindromes. Since $a_{p+3} = a_{s-p-2} = 1$, Lemma 2.26 implies that $a_{p+3} = \cdots = a_{s-p-2} = 1$, so $T = C(1, 0^{(p+1)}, 1^{(s-2p-4)}, 0^{(p+1)}, 1)$. By Proposition 2.30, again $\operatorname{drn}(T) \leq 2$. $\square$

2.6 General caterpillars

Having shown that $\mathrm{drn}(T) \le 2$ whenever T has the form $C(1, 0, a_3, \ldots, a_{s-2}, 0, 1)$, we may exclude such caterpillars (and stars) from our study of general caterpillars. In the general case, we will use the dacards obtained by deleting the first spine vertex v_1 and one of its leaf neighbors. This choice will determine T except in some cases. Again we must handle the exceptional cases separately, choosing a different pair of dacards. The next several propositions handle these cases. Note that setting $k = 0$ in the first would yield a path.

Proposition 2.32. *If $T = C(k+1, k^{(m)}, k+1)$, where $k, m \ge 1$, then $\mathrm{drn}(T) = 2$.*

Proof. The cards obtained by deleting leaf neighbors of v_1 and v_2 are $C(k^{(m+1)}, k+1)$ and $C(k+1, k-1, k^{(m-1)}, k+1)$. Let G be a reconstruction from these dacards, with u and v respectively being the added vertices of degree 1; G must be a tree. Since the endpoints of the spine in $G - v$ both have $k + 1$ leaf neighbors, G has two vertices at distance $m + 1$ that each have at least $k + 1$ leaf neighbors. Since $G - u$ has only one vertex with $k + 1$ leaf neighbors, the neighbor of u in $G - u$ must have distance $m + 1$ from the spine endpoint having $k + 1$ leaf neighbors. There is only one such vertex, so $G \cong C(k + 1, k^{(m)}, k + 1)$. □

A *branch vertex* is a vertex with degree at least 3. Let B_k denote the caterpillar formed by giving two leaf neighbors to one end of P_k. Let z_k denote the third leaf in B_k.

Proposition 2.33. *If $T = C(2, 0^{(s-2)}, 2)$, where $s \ge 3$, then $\mathrm{drn}(T) = 2$.*

Proof. Let $p = \lceil s/2 \rceil$. Note that v_p is centroidal in T and v_{p-1} is not. The cards C_1 and C_2 obtained by deleting v_p and v_{p-1} are $B_{p-1} + B_{s-p}$ and $B_{p-2} + B_{s-p+1}$, respectively. Let $D_1 = (C_1, 2)$ and $D_2 = (C_2, 2)$; these are the dacards for v_p and

v_{p-1} when $s \geq 5$. We postpone the special cases $s = 4$ and $s = 3$ (when $s = 2$, the caterpillar reduces to H_1).

Let G be a reconstruction from $\{D_1, D_2\}$, where $C_1 = G - u$ and $C_2 = G - v$. By Lemma 2.24, G is a tree, and each of u and v has one neighbor in each component of its dacard.

Case 1: $uv \in E(G)$. Since $d_G(u) = d_G(v) = 2$, vertex v is a leaf in $G - u$, and u is a leaf in $G - v$. Thus $G - v$ can be obtained from $G - u$ by deleting the leaf v in $G - u$ and attaching u to one vertex in the other component of $G - u$. Since $p - 2 < p - 1 \leq s - p < s - p + 1$, with the components of $G - u$ being isomorphic when $p - 1 = s - p$, obtaining a component of $G - v$ by deleting a leaf of a component of $G - u$ happens only by deleting z_{p-1} from B_{p-1} to obtain B_{p-2}. Hence B_{s-p+1} is the component of $G - v$ containing u, and it arises from B_{s-p} only by attaching u to z_{s-p}. Now $G \cong T$.

Case 2: $uv \notin E(G)$. Let Q and Q' be the components of $G - u$, with $v \in V(Q)$. Since $uv \notin E(G)$, we have $d_{G-u}(v) = 2$. Now v is a cut-vertex of Q. Let q be the order of the component of $Q - v$ not containing the neighbor of u in $V(Q)$. It follows that $G - v$ has components of orders q and $s + 3 - q$; we also know that these values are p and $s - p + 3$. Since the orders of Q and Q' differ by at most one, we have $q < s + 3 - q$. We conclude that $q = p$. To accommodate the inclusion of vertex v and another vertex, Q needs at least $p + 2$ vertices, so $Q = B_{s-p} \cong B_p$ (with s even), v is the vertex of B_{s-p} adjacent to z_{s-p}, and u is adjacent to z_{s-p}. Now examination of $G - v$ shows that the neighbor of u in B_{p-1} is z_{p-1}, and again $G \cong T$.

In either case, when $s \geq 5$, we conclude that $G \cong T$. For $s \in \{3, 4\}$, we again use dacards for v_p and v_{p-1}, but now $p = 2$, and we obtain $C_1 = P_3 + B_{s-p}$ and $C_2 = 2K_1 + B_{s-p+1}$, with $D_1 = (C_1, 2)$ and $D_2 = (C_2, 3)$. Although Lemma 2.24 does not apply, still every reconstruction G (with $C_1 = G - u$ and $C_2 = G - v$) is a tree. This holds because D_1 implies that G has no isolated vertex, and then

D_2 gives v a neighbor in each component of $G-v$.

If $s=3$, then $C_1=2P_3$, which yields $\Delta(G)\le 3$. Hence we cannot make v adjacent to the center of B_{s-p+1} (which equals $K_{1,3}$), and making it adjacent to a leaf of B_{s-p+1} yields $G\cong T$.

If $s=4$, then $T=C(2,0,0,2)$, with $C_1=P_3+K_{1,3}$ and $C_2=2K_1+B_3$. If v is adjacent to z_3 in the component B_3 of $G-v$, then $G\cong T$, so we exclude the other three possibilities. If G has a vertex x of degree 4, then $\Delta(G-u)=\Delta(G-v)=3$ requires $u,v\in N_G(x)$. Now x has a neighbor v of degree 3, but restoring u to $G-u$ gives x no neighbor with degree more than 3. Hence $\Delta(G)=3$. This requires u to be adjacent to the central vertex of P_3 and a leaf of $K_{1,3}$ in the two components of C_1, yielding $G\cong T$. □

Proposition 2.34. *If* $T=C(k+2,(0,k)^{(m)},0,k+2)$, *with* $k\ge 0$ *and* $m\ge 1$, *then* $\mathrm{drn}(T)\le 2$.

Proof. The case where $k=0$ is a special case of Proposition 2.33, so we may assume that $k\ge 1$. In that case T is unicentroidal and has a leaf adjacent to the centroid whose deletion leaves a unicentroidal subtree. By Proposition 2.23, $\mathrm{drn}(T)=2$. □

For a general caterpillar T, with $T=C(a_1,\ldots,a_s)$, we want to make a uniform choice of two dacards. The main lemma shows that this choice determines T unless T belongs to one of several exceptional classes of caterpillars. The proof of the theorem then uses the classes we have already discussed to handle the exceptional classes.

Lemma 2.35. *If* $T=C(a_1,\ldots,a_s)$, *then the dacards for an endpoint of the spine and one of its leaf neighbors determine* T *unless* T *is* Type t *for* $\mathrm{t}\in\{1,2,3,4\}$, *defined as follows:*

(1) $T=C(1,0,a_3,\ldots,a_s)$ *with* $s\ge 3$;

(2) $T = C(2, (0,0)^{(m)}, (1,0)^{(n)}, 2)$ *with* $m, n \geq 0$*;*

(3) $T = C(k+1, k^{(m)}, (k+1)^{(n)})$ *with* $k, m, n \geq 1$*;*

(4) $T = C(k+2, (0,k)^{(m)}, (0,k+1)^{(n)}, 0, k+2)$ *with* $k, n \geq 0$ *and* $m \geq 1$.

Proof. Since $\mathrm{drn}(K_{1,t}) = 1$, we may assume that $s \geq 2$. Let $\langle v_1, \ldots, v_s \rangle$ be the spine of T. Recall that $a_1, a_s \geq 1$. Specify the dacards by deleting v_1 and by deleting a leaf neighbor ℓ of v_1. Let $T_1 = T - \ell$, and let T_2 be the nontrivial component of $T - v_1$. Thus the dacards are $(a_1 K_1 + T_2, a_1 + 1)$ and $(T_1, 1)$. From the dacard $(T_1, 1)$, any reconstruction G is a tree. Define u and v by $G - u = T_1$ and $G - v = a_1 K_1 + T_2$. Let x the neighbor of u in G, and let y be the non-leaf neighbor of v in G (since G is a tree, $d_G(v) = a_1 + 1$ forces v to have one neighbor in T_2).

Define r and s by letting the spine of T_2 be $\langle v_r, \ldots, v_s \rangle$ and the spine of T_1 be $\langle v_q, \ldots, v_s \rangle$. We list four events; always (U1 or U2) and (V1 or V2) occurs. Note that if U1 and V1 occur, then T has Type 1, so we may assume that this case does not occur (and also that $G \ncong T$).

U1: $a_1 = 1, \quad q = 2, \quad \operatorname{diam} T_1 = s.$

U2: $a_1 > 1, \quad q = 1, \quad \operatorname{diam} T_1 = s + 1.$

V1: $a_2 = 0, \quad r = 3, \quad \operatorname{diam} T_2 = s - 1.$

V2: $a_2 > 0, \quad r = 2, \quad \operatorname{diam} T_2 = s.$

We call the descriptions of G obtained from $G - u$ and $G - v$ the *u-description* and the *v-description* of G. The cases depend on the location of y in T_2. Most importantly, this determines whether G is a caterpillar.

Case 1: y is in $\{v_{r+1}, \ldots, v_{s-1}\}$ or is a leaf neighbor of such a vertex. Since we make v (with its a_1 leaf neighbors) adjacent to y, in this case G is not a caterpillar.

The u-description also produces G and hence is not a caterpillar. Thus x is a leaf neighbor of a vertex in $\{v_{q+1}, \ldots, v_{s-1}\}$. The skeleton G' has three leaves. In the v-description, the leaves are v_r, v_s, and v; also, G' has $s-r+1$ edges if y is in the spine of T_2, otherwise $s-r+2$. In the u-description, the leaves of G' are v_q, v_s and x, and G' has $s-q+1$ edges.

Let S_u and S_v denote the multiset of degrees in G of the leaves of G' under the u-description and v-description of G, respectively. Equating the numbers of edges of G' in the two descriptions yields several possibilities.

(i) If $q=1$, then $r=2$ and y is not in the spine of T_2. Now $S_u = \{a_1, a_s+1, 2\}$ and $S_v = \{a_2+1, a_s+1, a_1+1\}$. Equality requires $a_1 = 1$, which contradicts $q=1$.

(ii) If $q=2$, then $r=2$, since otherwise T has Type 1. Now $S_u = \{a_2+2, a_s+1, 2\}$ and $S_v = \{a_2+1, a_s+1, 2\}$, and equality cannot hold.

Case 2: y is a leaf neighbor of v_r or v_s. For such y, if $a_2 = 0$ and hence $r=3$, then $G \cong T$ or $G = C(a_3+1, a_4, \ldots, a_{s-1}, a_s-1, 0, a_1)$. If $a_2 > 0$ and hence $r=2$, then $G = C(a_1, 0, a_2-1, a_3, \ldots, a_s)$ or $G = C(a_2, \ldots, a_{s-1}, a_s-1, 0, a_1)$.

Subcase 2a: $a_2 = 0$. Here $G = C(a_3+1, a_4, \ldots, a_{s-1}, a_s-1, 0, a_1)$. Avoiding Type 1 requires $a_1 > 1$ and $q=1$, so $\operatorname{diam} T_1 = s+1 = \operatorname{diam} G$. Since G is a caterpillar with diameter $\operatorname{diam} T_1$, vertex x is on the spine of T_1, say $x = v_j$. With $G \not\cong T$, we have $j > 1$, and the u-description is $G = C(a_1-1, a_2, \ldots, a_{j-1}, a_j+1, a_{j+1}, \ldots, a_s)$, with $a_2 = 0$.

In obtaining the multiset of leaf degrees for G from that of T, in both the v-description and the u-description one term increases and one term decreases. The values that change must be the same in each instance; hence $a_1 = a_s$ and $a_3 = a_j$. Since $a_1 \neq a_1 - 1$, the descriptions match without reversal. Since $a_s = a_1 = a_3 + 2 = a_j + 2 > a_j + 1 > 0$, we conclude that $j \leq s-2$. Since $0 = a_{s-1} = a_{s-3} = \cdots$, we conclude that $s-j$ is even (otherwise $a_j + 1 = 0$).

If j and s are even, then $0 = a_2 = a_4 = \cdots = a_j = a_3$, so $a_1 = a_s = 2$. Also $0 = a_3 = \cdots = a_{s-1}$. Since $a_j + 1 = a_{j+2} = \cdots = a_{s-2} = a_s - 1 = 1$, we find that T is Type 2.

If j and s are odd, then $0 = a_2 = \cdots a_{s-1}$ and $a_1 - 2 = a_3 = \cdots = a_j$ and $a_j + 1 = a_{j+2} = \cdots = a_{s-2} = a_s - 1$, with $j \geq 3$. Letting $k = a_3$, we find that T is Type 4.

Subcase 2b: $a_2 > 0$. Here $\operatorname{diam} T_2 = s$, and hence $\operatorname{diam} G = s + 2$. Since adding u to T_1 can only add 1 to the diameter, $\operatorname{diam} T_1 = s + 1$, and hence $q = 1$, $a_1 > 1$, and x is a leaf neighbor of v_1 or v_s. Now the u-description is $G = C(1, a_1 - 2, a_2, \ldots, a_s)$ or $G = C(a_1 - 1, a_2, \ldots, a_{s-1}, a_s - 1, 1)$.

In both possibilities for the v-description with $a_2 > 0$, one end of the spine of G has a_1 leaf neighbors. Since $a_1 \notin \{1, a_1 - 1\}$, the second possibility for the u-description is forbidden. Furthermore, since $a_1 > 1$, the first possibility must be oriented so that a_s in the u-description matches up with a_1 in the v-description. We have two choices.

(i) $(a_s, \ldots, a_3, a_2 - 1, 0, a_1) = (1, a_1 - 2, a_2, \ldots, a_s)$. This is forbidden, since it requires $1 = a_s = a_1$, but $a_1 > 1$.

(ii) $(a_2, \ldots, a_{s-1}, a_s - 1, 0, a_1) = (1, a_1 - 2, a_2, \ldots, a_s)$. Since $0 = a_{s-1} = a_{s-3} = \cdots$ and $1 = a_2 = a_4 = \cdots$, we conclude that s is even. Now $a_1 - 2 = 0$ and $a_s - 1 = 1$, and T is Type 2 with $m = 0$.

Case 3: $y \in \{v_r, v_s\}$. If $y = v_r$, then $G \cong T$, so we may assume $y = v_s$ and G is a caterpillar with diameter $s - r + 3$. Since G is a caterpillar, x is a spine vertex of T_1 or a leaf neighbor of v_q or v_s.

If x is a leaf neighbor of v_q or v_s, then adding u to T_1 enlarges the diameter, so $\operatorname{diam} G = s - q + 3$. Hence $q = r$, which requires $a_1 = 1$ and $a_2 > 0$, and $q = r = 2$. Since $a_1 = 1$, setting x to a leaf neighbor of v_2 yields $G \cong T$. Hence the v-description is $G = C(a_2, \ldots, a_s, a_1)$ and the u-description is $G =$

$C(a_2+1, a_3, \ldots, a_{s-1}, a_s-1, 1)$. Since $a_2 > 0$, the descriptions must match up without reversal, which fails because $a_2 \neq a_2+1$.

Finally, we may assume that x is a spine vertex v_j in T_1. Now $\operatorname{diam} G = s-q+2$, so $q = r-1$. Avoiding Type 1 leaves only $q = r-1 = 1$, so $a_1 > 1$ and $a_2 > 0$. If $j = 1$, then $G \cong T$, so $j > 1$. Now the v-description is $G = C(a_2, \ldots, a_s, a_1)$ and the u-description is $G = C(a_1-1, a_2, \ldots, a_{j-1}, a_j+1, a_{j+1}, \ldots, a_s)$. Since $a_1 - 1 \neq a_1$, the descriptions must match up without reversal. Two posibilities remain.

(i) If $j = s$, then matching positions yields $a_1 - 1 = a_2 = \cdots = a_s$. Now the v-description of G is the reverse of the original description of T, and hence $G \cong T$.

(ii) If $1 < j < s$, then matching positions yields $a_1 - 1 = a_2 = \cdots = a_j = a_{j+1} - 1 = \cdots = a_s - 1$. Letting $a_2 = k$, we have $T = C(k+1, k^{(j-1)}, (k+1)^{(s-j)})$. We may assume that $k \geq 1$, since otherwise T is Type 1. Now T is Type 3. □

Theorem 2.36. *If T is a caterpillar that is neither H_1 nor a star, then* $\operatorname{drn}(T) = 2$.

Proof. Let $T = C(a_1, \ldots, a_s)$. As in Section 2.5, reversing the order of the spine vertices does not change the isomorphism class of a caterpillar; $T \cong T'$, where $T' = C(a_s, \ldots, a_1)$. In Lemma 2.35 we used dacards corresponding to the first spine endpoint and a leaf adjacent to it, but similar results hold by taking dacards corresponding to the *last* spine vertex and a leaf adjacent to it. Thus our choice of two dacards, from one end of T or the other, uniquely determines T unless both T and T' have a Type listed in Lemma 2.35.

Suppose first that T is Type 1. Suppose that T' is Type 1 as well. We have $T = C(1, 0, a_3, \ldots, a_{s-2}, 0, 1)$, and $\operatorname{drn}(T) \leq 2$ by Proposition 2.31. Since all other Types end with $a_s > 1$, but $a_1 = 1$, the reversal of a Type 1 caterpillar cannot be Type 2, 3, or 4. This completes the proof when T (or T') is Type 1.

Suppose next that T is Type 2. Since the length of the spine has different parity in Type 2 and Type 4, T' is not of Type 4. If T' is Type 2 or Type 3, then either $T = C(2, 2)$ and $T \cong H_1$, or $T = C(2, (0, 0)^{(m)}, 2)$ with $m \geq 1$, in which case $\mathrm{drn}(T) \leq 2$ by Proposition 2.33.

If T and T' are both Type 3, then $T = C(k + 1, k^{(m)}, k + 1)$ with $k, m \geq 1$, and $\mathrm{drn}(T) \leq 2$ by Proposition 2.32. Since the entries in specifying a Type 3 caterpillar are all positive, and for Type 4 they are not, T and T' cannot be Type 3 and Type 4.

Finally, if T and T' are both Type 4, then $n = 0$. Now $\mathrm{drn}(T) \leq 2$ by Proposition 2.34.

Having exhausted all cases, the proof is complete. □

There is hope to complete a proof that $\mathrm{drn}(T) \leq 2$ for all but finitely many trees. Building upon our result, one can try to make a choice of two dacards that determines T when T is not a caterpillar, with finitely many exceptions. As happened in the proofs of our results on caterpillars, there may be several special classes in addition to caterpillars where the dacards needs to be chosen in other ways.

CHAPTER 3

Degree-sequence-forcing sets

3.1 Introduction

A graph class $\mathcal{C}$ is *hereditary* if whenever G is an element of $\mathcal{C}$, every induced subgraph of G also belongs to $\mathcal{C}$. Recall that, given a set $\mathcal{F}$ of graphs, G is *$\mathcal{F}$-free* if no induced subgraph of G is isomorphic to an element of $\mathcal{F}$.

Hereditary classes are exactly those consisting of the $\mathcal{F}$-free graphs for some set $\mathcal{F}$ of graphs. Many important classes of graphs are hereditary, and several celebrated theorems have specified the minimal forbidden subgraphs for these classes. For example, Kuratowski's Theorem [31] can be reformulated as a statement of which induced subgraphs are forbidden for planar graphs, and the Strong Perfect Graph Theorem [12] characterizes perfect graphs in terms of their forbidden subgraphs.

We say that a graph class $\mathcal{C}$ is *degree-determined*, or that it has a *degree sequence characterization*, if it is possible to determine whether a graph G belongs to $\mathcal{C}$ from just the degree sequence of G. Most graph classes of interest do not have degree sequence characterizations, but they are useful when they do exist, as they often lead to very efficient algorithms for recognizing membership in a degree-determined class of graphs.

In this chapter, we address the question of which classes of graphs can be characterized both in terms of their degree sequences and in terms of a set of forbidden subgraphs. More precisely, we make the following definition.

Definition 3.1. A set $\mathcal{F}$ of graphs is *degree-sequence-forcing* if whenever some realization of a graphic sequence π is $\mathcal{F}$-free, every other realization of π is $\mathcal{F}$-free as well.

We seek to characterize degree-sequence-forcing sets of graphs. Our interest in degree-sequence-forcing sets is motivated in part by the class of *split graphs*, those whose vertex sets can be partitioned into a clique and an independent set. Split graphs have the following two characterizations:

Theorem 3.2 (Földes–Hammer [16]). *A graph is split if and only if it is $\{2K_2, C_4, C_5\}$-free.*

Theorem 3.3 (Hammer–Simeone [20]). *If G is a graph having degree sequence $d(G) = (d_1, \ldots, d_n)$ in nonincreasing order, then G is split if and only if*

$$\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i,$$

where $m = \max\{k \,:\, d_k \geq k-1\}$.

From these two theorems, $\{2K_2, C_4, C_5\}$ is a degree-sequence-forcing set. Other examples of degree-determined hereditary families have appeared in the literature; Table 3.1 lists several. The sets of graphs in the rightmost column of the table are all degree-sequence-forcing sets.

We begin our analysis in Section 3.2 by proving several necessary and sufficient conditions for a set of graphs to be degree-sequence-forcing. We then use these results in Section 3.3 to characterize all degree-sequence-forcing sets with size at most 2. A degree-sequence-forcing set is *minimal* if it contains no proper degree-sequence-forcing subset. In Section 3.4 to determine all non-minimal degree-sequence-forcing sets with size 3. In Section 3.5 we study minimal degree-sequence-forcing sets and give a brief discussion of their properties. We

Class	Characterizations		List of forbidden subgraphs
	Degree sequence	Forbidden subgraph	
Split graphs	[20]	[16]	$\{2K_2, C_4, C_5\}$
Threshold graphs	[19]	[14]	$\{2K_2, C_4, P_4\}$
Pseudo-split graphs	[36]	[36]	$\{2K_2, C_4\}$
Matroidal graphs	[37, 50]	[42]	$\{C_5$, $K_2 + P_3$, $2K_1 \vee (K_2 + K_1)$, P_5, house, chair, kite, $K_2 + K_3$, $K_{2,3}$, 4-pan, co-4-pan$\}$
Matrogenic graphs	[37, 50]	[15]	$\{K_2 + P_3$, $2K_1 \vee (K_2 + K_1)$, P_5, house, chair, kite, $K_2 + K_3$, $K_{2,3}$, 4-pan, co-4-pan$\}$

Table 3.1: Graph classes characterized by both degree sequences and forbidden subgraphs.

conclude the chapter in Section 3.6 by discussing *edit-leveling* sets of graphs, which are degree-sequence-forcing sets satisfying a much stronger condition in terms of degree sequences.

3.2 Conditions on degree-sequence-forcing sets

In this section we provide some necessary and some sufficient conditions for a set of graphs to be degree-sequence-forcing. We first show how degree-sequence-forcing sets may be used to give rise to other degree-sequence-forcing sets.

Proposition 3.4. *Given a set $\mathcal{G}$ of graphs, let $\mathcal{F}$ be the set of minimal elements of $\mathcal{G}$ under the induced subgraph relation. The set $\mathcal{G}$ is degree-sequence-forcing if and only if $\mathcal{F}$ is degree-sequence-forcing.*

Proof. It is easy to see that a graph is $\mathcal{G}$-free if and only if it is $\mathcal{F}$-free. If either the $\mathcal{G}$-free or the $\mathcal{F}$-free graphs may be recognized by their degree sequences, then membership in the other set of graphs is recognizable from the degree sequence as well. □

Proposition 3.5. *The union of degree-sequence-forcing sets is degree-sequence-forcing.*

Proof. Let $\mathcal{I}$ be an index set, and let $\mathcal{F}_i$ be a degree-sequence-forcing set for each $i \in \mathcal{I}$. Define $\mathcal{F} = \bigcup_{i\in\mathcal{I}} \mathcal{F}_i$. Suppose that π is a graphic list having a realization that induces an element F of $\mathcal{F}$. The graph F belongs to $\mathcal{F}_j$ for some $j \in \mathcal{I}$, and since $\mathcal{F}_j$ is degree-sequence-forcing, every realization of π induces an element of $\mathcal{F}_j$ and hence an element of $\mathcal{F}$. Thus $\mathcal{F}$ is degree-sequence-forcing. $\square$

Proposition 3.6. *If $\mathcal{F}$ is a degree-sequence-forcing set of graphs, then $\mathcal{F}^c$ is also degree-sequence-forcing, where $\mathcal{F}^c = \{\overline{G} : G \in \mathcal{F}\}$.*

Proof. Let $\mathcal{F}$ be degree-sequence-forcing, and suppose that π is a graphic list having a realization G that induces an element F of $\mathcal{F}^c$. The graph $\overline{G}$ induces $\overline{F}$, an element of the degree-sequence-forcing set $\mathcal{F}$, so every realization of the degree sequence of $\overline{G}$ induces an element of $\mathcal{F}$. It follows that every realization of π induces an element of $\mathcal{F}^c$, so $\mathcal{F}^c$ is degree-sequence-forcing. $\square$

A *unigraph* is a graph that is the unique (unlabeled) realization of its degree sequence. The following remark is an easy consequence of the definition of a degree-sequence-forcing set and establishes a sufficient condition for a set of graphs to be degree-sequence-forcing.

Remark 3.7. *Let $\mathcal{F}$ be a set of graphs. If every $\mathcal{F}$-free graph is a unigraph, then $\mathcal{F}$ is degree-sequence-forcing.*

A 2-*switch* is an operation on a graph G that deletes two disjoint edges uv and xy such that $ux, vy \notin E(G)$ and adds ux and vy to the graph. We denote such a 2-switch by $\{uv, xy\} \rightrightarrows \{ux, vy\}$. This operation is important in the study of degree sequences because of the following result.

Theorem 3.8 (Fulkerson et al. [17]). *Graphs H and H' on the same vertex set have $d_H(v) = d_{H'}(v)$ for every vertex v if and only if H' can be obtained by performing a finite sequence of 2-switches on H.*

Definition 3.9. Given a set $\mathcal{F}$ of graphs and an element F of $\mathcal{F}$, the graph F *switches to* $\mathcal{F} - F$ if whenever H and H' are two graphs such that H induces F while H' is F-free, and H' can be obtained by performing a single 2-switch on H, the graph H' induces an element of $\mathcal{F} - F$.

If $\mathcal{F}$ consists of two graphs F and G, instead of saying that F switches to $\{G\}$, we say simply that F switches to G.

Proposition 3.10. *Let $\mathcal{F}$ be a set of graphs. The following statements are equivalent.*

(i) *If $F \in \mathcal{F}$, then F switches to $\mathcal{F} - F$.*

(ii) *For every $F \in \mathcal{F}$, if H and H' are any two graphs having the same degree sequence such that H induces F and H' is F-free, then H' induces an element of $\mathcal{F} - F$.*

(iii) *$\mathcal{F}$ is a degree-sequence-forcing set.*

Proof. (i) $\Longrightarrow$ (ii): Let F be an element of $\mathcal{F}$. Suppose that H and H' are two graphs with the same degree sequence such that H induces F and H' is F-free. By Theorem 3.8, there exists a finite sequence of 2-switches that, when applied to H, produces H'. Let H_i denote the graph obtained after the ith 2-switch in this sequence, so that $H_0 = H$ and $H_k = H'$. If j is the first index such that H_{j+1} does not induce F, then H_j induces F. Since F switches to $\mathcal{F} - F$, the graph H_{j+1} must induce an element F' of $\mathcal{F} - F$. If H' does not induce F', then there exists a least index j' exceeding j such that $H_{j'+1}$ does not induce F'; since F' switches to $\mathcal{F} - F'$, we conclude that $H_{j'}$ induces an element of $\mathcal{F} - F'$. By induction on i, we see that each H_i induces an element of $\mathcal{F}$; if H_k does not induce F, then it induces an element of $\mathcal{F} - F$.

(ii) $\implies$ (iii): Suppose that π is a graphic sequence having a realization H that induces an element F of $\mathcal{F}$. If H' is any other realization of π, then by (ii) H' induces either F or an element of $\mathcal{F} - F$; in either case H' induces an element of $\mathcal{F}$.

(iii) $\implies$ (i): Let $\mathcal{F}$ be a degree-sequence-forcing set of graphs, and let F be an element of $\mathcal{F}$. Suppose that H and H' are two graphs such that H induces F while H' is F-free, and H' can be obtained by performing a single 2-switch on H. Since $\mathcal{F}$ is degree-sequence-forcing and the degree sequence of H has a realization that induces an element of $\mathcal{F}$, we conclude that H' induces an element of $\mathcal{F}$ and hence of $\mathcal{F} - F$. ☐

For sets $\mathcal{F}$ with more than a few graphs, using Proposition 3.10 to show that $\mathcal{F}$ is degree-sequence-forcing can be quite cumbersome in practice. However, this proposition will be important in characterizing degree-sequence-forcing pairs in Section 3.3. We define an ordered pair (H, H') of graphs to be *$\mathcal{F}$-breaking* if their degree sequences are the same, H induces an element of $\mathcal{F}$, and H' is $\mathcal{F}$-free. By Proposition 3.10, a set $\mathcal{F}$ is degree-sequence-forcing if and only if no $\mathcal{F}$-breaking pair exists. In fact, a stronger result holds, as we show in the following result.

Proposition 3.11. *If $\mathcal{G}$ is a set of graphs that is not degree-sequence-forcing, then there exists a $\mathcal{G}$-breaking pair (H, H') such that H and H' each have at most $|V(G)| + 2$ vertices, where G is a graph in $\mathcal{G}$ with the most vertices.*

Proof. Since $\mathcal{G}$ is not degree-sequence-forcing, by Proposition 3.10 there exists a $\mathcal{G}$-breaking pair (J, J') of graphs. By Theorem 3.8, there exists a sequence $J = J_0, J_1, J_2, \ldots, J_k = J'$ of graphs in which J_i is obtained via a 2-switch on

J_{i-1} for $i \in \{1, \dots, k\}$. Define ℓ to be the largest index such that J_ℓ induces an element G of $\mathcal{G}$, so that $(J_\ell, J_{\ell+1})$ is a $\mathcal{G}$-breaking pair. Let V denote the vertex set of an induced copy of G in J_ℓ, and let W denote the set of 4 vertices involved in the 2-switch transforming J_ℓ into $J_{\ell+1}$. Since G is not induced on V in $J_{\ell+1}$, the 2-switch performed must add an edge to or delete an edge from $J_\ell[V]$; hence $|W \cap V| \geq 2$ and $|V \cup W| \leq |V| + 2$. Thus $(J_\ell[V \cup W], J_{\ell+1}[V \cup W])$ is a $\mathcal{G}$-breaking pair on at most $|V(G)| + 2$ vertices. Taking G to be a graph in $\mathcal{G}$ with the most vertices yields the result. □

We now provide a number of necessary conditions on a degree-sequence-forcing set by considering the effect that 2-switches can have on certain graph parameters.

Proposition 3.12. *Every degree-sequence-forcing set contains a forest in which each component is a star.*

Proof. Let $\mathcal{F}$ be a set containing no forest, and let $F \in \mathcal{F}$ be a graph having the minimum number of cycles among graphs in $\mathcal{F}$. Let xy be an edge of a cycle in F. Form H by adding to F two new vertices u and v and the edge uv. Form H' from H via the 2-switch $\{uv, xy\} \rightrightarrows \{ux, vy\}$. The graph H' has fewer cycles than F and hence is $\mathcal{F}$-free; thus $\mathcal{F}$ is not degree-sequence-forcing.

Having shown that every degree-sequence-forcing set contains a forest, let $\mathcal{F}$ be a degree-sequence-forcing set. Suppose that every forest in $\mathcal{F}$ has a component of diameter at least 3 (and hence is not a forest of stars). Among the forests in $\mathcal{F}$, consider those which minimize the length of a longest path, and among these latter forests choose F having a minimum number of paths of this length. Let ℓ denote the maximum length of a path in F, and let xy be an internal edge of a path in F of length ℓ. Form a graph H by adding to F two new vertices u and v and the edge uv. Form H' from H via the 2-switch $\{uv, xy\} \rightrightarrows \{ux, vy\}$. Now H' is a forest having fewer paths of length ℓ than F does, and the longest path in

H' has length at most ℓ. It follows that (H, H') is an $\mathcal{F}$-breaking pair of graphs. This is a contradiction, since $\mathcal{F}$ is degree-sequence-forcing. Thus, $\mathcal{F}$ contains a forest in which every component is a star. $\square$

We may generalize the approach of proposition. Define a graph parameter to be *order-preserving* if $p(G) \le p(H)$ whenever G is an induced subgraph of H.

Remark 3.13. *Let $p(G)$ be an order-preserving parameter and c a constant. Suppose that for every graph G such that $p(G) > c$, there exist graphs H and H' such that the graph H contains G as an induced subgraph, H' is obtained by performing a 2-switch on H, and $p(H') < p(G)$. Every degree-sequence-forcing set contains an element F such that $p(F) \le c$.*

The first paragraph of the proof of Proposition 3.12 illustrates this idea; there the parameter $p(G)$ is the number of cycles in G, and $c = 0$. The conclusion of Remark 3.13 also holds when $p(G)$ takes values in any linearly ordered set, and such a formulation could be used to provide an alternate version of the second paragraph of the proof of Proposition 3.12.

Corollary 3.14. *Every degree-sequence-forcing set contains a graph that is the complement of a forest of stars.*

Proof. Let $\mathcal{F}$ be a degree-sequence-forcing set. By Proposition 3.6, the set $\mathcal{F}^c$ is degree-sequence-forcing and hence contains a forest of stars by Proposition 3.12. Thus $\mathcal{F}$ contains the complement of a forest of stars. $\square$

Proposition 3.15. *Every degree-sequence-forcing set contains a graph that is a disjoint union of complete graphs.*

Proof. Let $p(G)$ denote the minimum number of edges that need to be added to G to make every component a complete subgraph. Note that $p(G)$ is an order-preserving parameter, as deleting any vertex of G cannot increase the number of non-adjacent pairs of vertices in the same component. Let G be an arbitrary graph such that $p(G) \geq 1$, and let x and y be two non-adjacent vertices in a component of G. Form a graph H by adding to G two new vertices u and v and edges ux and vy. Form H' from H via the 2-switch $\{ux, vy\} \rightrightarrows \{uv, xy\}$. Note that $p(H') < p(G)$. By Remark 3.13, if $\mathcal{F}$ is a degree-sequence-forcing set, then $\mathcal{F}$ contains an element F such that $p(F) = 0$, that is, F is a disjoint union of complete graphs. □

Corollary 3.16. *Every degree-sequence-forcing set contains a complete multipartite graph.*

We have shown that every degree-sequence-forcing set contains at least one element from each of several classes, which we denote as follows:

$$\begin{aligned}
\mathbb{K} &= \{\text{disjoint unions of complete graphs}\},\\
\mathbb{K}^c &= \{\text{complete multipartite graphs}\},\\
\mathbb{S} &= \{\text{forests of stars}\},\\
\mathbb{S}^c &= \{\text{complements of forests of stars}\}.
\end{aligned}$$

3.3 Singletons and pairs

In this section we use the results of the previous section to completely determine all degree-sequence-forcing sets of size at most 2. We immediately determine the degree-sequence-forcing singleton sets.

Theorem 3.17. *A singleton set $\{F\}$ is degree-sequence-forcing if and only if $F \in \{K_1, K_2, 2K_1\}$.*

Proof. We have assumed that all graphs have at least one vertex, so the statement that $\{K_1\}$ is degree-sequence-forcing is vacuously true. A graph is $\{K_2\}$-free if and only if it is edgeless, which happens if and only if its degree sequence contains only zeros. Thus $\{K_2\}$ is degree-sequence-forcing, and by Proposition 3.6 the set $\{2K_1\}$ is degree-sequence-forcing as well.

Note now that if $\{F\}$ is a degree-sequence-forcing set, then F belongs to each of $\mathbb{K}$, $\mathbb{K}^c$, $\mathbb{S}$, and $\mathbb{S}^c$. Since every component of F is complete, F cannot induce P_3. As F is a complete multipartite graph, this means that either F has only one partite set, or every partite set contains only one vertex. Thus F is either K_n or nK_1 for some n. Since F is both a forest and the complement of a forest, we have $n \leq 2$, so $F \in \{K_1, K_2, 2K_1\}$. $\square$

We devote the rest of the section to proving the following result.

Theorem 3.18. *A pair of graphs comprises a degree-sequence-forcing set if and only if it is one of the following:*

(i) $\{A, B\}$, *where* $A \in \{K_1, K_2, 2K_1\}$ *and* B *is any graph;*

(ii) $\{P_3, K_3\}$, $\{P_3, K_3 + K_1\}$, $\{P_3, K_3 + K_2\}$, $\{P_3, 2K_2\}$, $\{P_3, K_2 + K_1\}$*;*

(iii) $\{K_2 + K_1, 3K_1\}$, $\{K_2 + K_1, K_{1,3}\}$, $\{K_2 + K_1, K_{2,3}\}$, $\{K_2 + K_1, C_4\}$*;*

(iv) $\{K_3, 3K_1\}$*;*

(v) $\{2K_2, C_4\}$.

We first show that these pairs are degree-sequence-forcing, after which we will show that no other pairs are degree-sequence-forcing. We recall from Chapter 1 that an alternating 4-cycle in a graph G is a configuration on four vertices of G in which two edges and two non-edges alternate in a cyclic fashion. An alternating 4-cycle and the minimal unlabeled subgraphs having an alternating 4-cycle are shown in Figure 3.1; these subgraphs are $2K_2$, C_4, and P_4. (Here and throughout this thesis, dotted segments will denote non-adjacencies.)

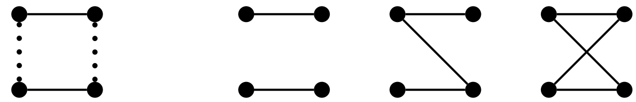


Figure 3.1: An alternating 4-cycle and the 4-vertex subgraphs in which it appears.

The $\{2K_2, C_4\}$-free graphs appear in Table 3.1; these graphs are called the *pseudo-split* graphs, and they were shown in [36] to form a degree-determined family. Thus $\{2K_2, C_4\}$ is degree-sequence-forcing. Each of the other pairs is degree-sequence-forcing by Remark 3.7 and the following.

Proposition 3.19. *If $\mathcal{F}$ is any of the following sets, then the $\mathcal{F}$-free graphs are all unigraphs:*

(i) $\{A, B\}$, *where* $A \in \{K_1, K_2, 2K_1\}$ *and* B *is any graph;*

(ii) $\{P_3, K_3\}$, $\{P_3, K_3 + K_1\}$, $\{P_3, K_3 + K_2\}$, $\{P_3, 2K_2\}$, $\{P_3, K_2 + K_1\}$;

(iii) $\{K_2 + K_1, 3K_1\}$, $\{K_2 + K_1, K_{1,3}\}$, $\{K_2 + K_1, K_{2,3}\}$, $\{K_2 + K_1, C_4\}$;

(iv) $\{K_3, 3K_1\}$.

Proof. (i) The $\{A, B\}$-free graphs form a subset of the $\{A\}$-free graphs, so it suffices to show that the $\{A\}$-free graphs are unigraphs. Since an alternating 4-cycle is required to perform a 2-switch, and alternating 4-cycles induce K_1, K_2, and $2K_1$, Theorem 3.8 implies that the $\{A\}$-free graphs are unigraphs.

(ii) Let G be a $\{P_3, K_3 + K_2\}$-free graph. A graph is $\{P_3\}$-free if and only if it is a disjoint union of complete graphs. Since G is additionally $\{K_3 + K_2\}$-free, either G induces K_3, in which case G has the form $K_n + mK_1$, or G is triangle free, in which case G has the form $mK_2 + nK_1$. In the first case no 2-switches are possible on G, and in the second case every 2-switch on G yields a graph isomorphic to G; hence G is a unigraph. If $\mathcal{F}$ is any of the sets listed in (ii) then the $\mathcal{F}$-free graphs form a subset of the $\{P_3, K_3 + K_2\}$-free graphs and hence are unigraphs.

(iii) Let $\mathcal{F}$ be a set listed in (iii). Suppose that G and G' are $\mathcal{F}$-free realizations of the same degree sequence. The graphs $\overline{G}$ and $\overline{G'}$ are both $\mathcal{F}^c$-free and have the same degree sequence. Since $\mathcal{F}^c$ appears in (ii), $\overline{G} \cong \overline{G'}$. It follows that $G \cong G'$, so the $\mathcal{F}$-free graphs are unigraphs.

(iv) A well-known elementary result of Ramsey Theory states that each of the $\{K_3, 3K_1\}$-free graphs contains at most five vertices, and direct verification shows that each $\{K_3, 3K_1\}$-free graph is a unigraph. □

To show that no pairs other than those listed in Theorem 3.18 are degree-sequence-forcing, we begin by employing "sieve" arguments: Given a graph A, we determine possible candidates for the graph B in a degree-sequence-forcing set $\{A, B\}$ by finding $\{A\}$-breaking pairs (H, H') of graphs. Proposition 3.10 implies that B is an induced subgraph of every graph H' such that (H, H') is an $\{A\}$-breaking pair for some H. Therefore, the graphs that appear in every such H' are the only possible choices for B.

Proposition 3.20. *Other than the pairs listed in Theorem 3.18, there are no degree-sequence-forcing pairs in which one graph has* 3 *or fewer vertices.*

Proof. Let $\mathcal{F} = \{A, B\}$, and suppose that $\mathcal{F}$ is degree-sequence-forcing. We must show that $\mathcal{F}$ is one of the sets listed in Theorem 3.18. This is clearly the case if A or B has fewer than 3 vertices.

Suppose that $\mathcal{F} = \{K_3, B\}$. Let $H_1 = K_3 + K_2$ and $H_1' = P_5$; further let H_2 be the house graph (that is, the complement of P_5), and let $H_2' = K_{2,3}$. Both (H_1, H_1') and (H_2, H_2') are $\{K_3\}$-breaking pairs. Since $\mathcal{F}$ is degree-sequence-forcing, B is a common induced subgraph of P_5 and $K_{2,3}$. The only such subgraphs on 3 or more vertices are P_3 and $3K_1$; hence $B \in \{K_1, K_2, 2K_1, P_3, 3K_1\}$, and $\mathcal{F}$ is listed

H_2 $\qquad$ H_2'

Figure 3.2: The graphs H_2 and H_2' from Proposition 3.21.

H_3 $\qquad$ H_3'

Figure 3.3: The graphs H_3 and H_3' from Proposition 3.21.

in Theorem 3.18.

If $\mathcal{F} = \{3K_1, B\}$, then Proposition 3.6 and the previous paragraph show that $B \in \{K_1, 2K_1, K_2, K_2 + K_1, K_3\}$. Hence $\mathcal{F}$ is listed in Theorem 3.18.

Suppose now that $\mathcal{F} = \{P_3, B\}$. Since $(P_5, K_3 + K_2)$ is a $\{P_3\}$-breaking pair, B is induced in $K_3 + K_2$, and hence $\mathcal{F}$ is one of the sets listed in Theorem 3.18.

Finally, suppose that $\mathcal{F} = \{K_2 + K_1, B\}$. Proposition 3.6 and the previous paragraph imply that B is induced in $K_{2,3}$, and hence $\mathcal{F}$ appears in the list in Theorem 3.18. □

We now consider pairs $\{A, B\}$ such that both graphs have at least four vertices. The *k-pan* is the graph consisting of a k-cycle plus a pendant edge. The *co-k-pan* is its complement.

Proposition 3.21. *The set $\{2K_2, C_4\}$ is the only degree-sequence-forcing pair of the form $\{2K_2, B\}$ or $\{C_4, B\}$ such that B contains at least 4 vertices.*

Proof. Let H_1 be the co-4-pan, and let H_1' be the 4-pan. Let H_2 and H_2' be the graphs shown in Figure 3.2, and let H_3 and H_3' be the graphs shown in Figure 3.3. The ordered pairs (H_1, H_1'), (H_2, H_2'), and (H_3, H_3') are all $\{2K_2\}$-breaking, so if $\{2K_2, B\}$ is degree-sequence-forcing, then B must be induced in each of H_1',

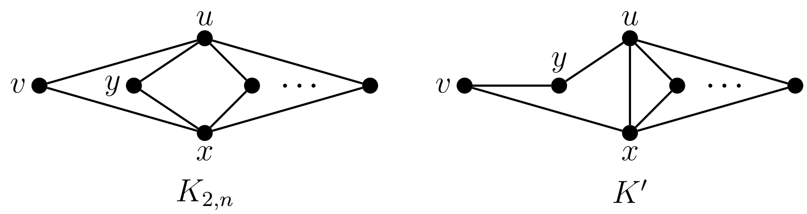


Figure 3.4: The graphs $K_{2,n}$ and K' from Lemmas 3.22 and 3.24.

H_2', and H_3'. The only induced subgraph with at least 4 vertices common to H_1', H_2', and H_3' is C_4, so $B = C_4$. From Proposition 3.6 it follows that if $\{C_4, B\}$ is degree-sequence-forcing and B has at least four vertices, then $B = 2K_2$. □

Lemma 3.22. *For $n \geq 4$, the graph nK_1 switches to B if and only if B is an induced subgraph of $K_2 + (n-2)K_1$.*

Proof. Let H be a graph inducing nK_1, and let U be a collection of n pairwise nonadjacent vertices in H. Let H' be an $\{nK_1\}$-free graph obtained by performing a single 2-switch on H. Such a 2-switch must place an edge between two vertices of U, and such a 2-switch can involve at most two vertices of U. In H', the subgraph induced on U is isomorphic to $K_2 + (n-2)K_1$, so nK_1 switches to every induced subgraph of this graph.

We now show that every graph B to which nK_1 switches is an induced subgraph of $K_2 + (n-2)K_1$. Define $H_1 = K_{2,n}$, and let the cycle $[u, y, x, v]$ be any 4-cycle in the graph, as shown in Figure 3.4. Form K' via the 2-switch $\{uv, xy\} \rightrightarrows \{ux, vy\}$, and let $H_1' = K$. Let $H_2 = P_5 + (n-3)K_1$ and $H_2' = K_2 + K_3 + (n-3)K_1$; also let $H_3 = 2P_3 + (n-4)K_1$ and $H_3' = P_4 + K_2 + (n-4)K_1$.

Observe that graphs H_1, H_2, and H_3 all induce nK_1. If nK_1 switches to B, then B is induced in each of H_1', H_2', and H_3', because none of these graphs induce nK_1. Since B is induced in H_2', it is a disjoint union of at most $n-1$ complete graphs. Each component of B has either one or two vertices, since H_3' induces no triangle. Furthermore, since B is induced in H_1', it does not induce $2K_2$ and

Figure 3.5: The graphs H_2 and H_2' from Lemma 3.24.

hence has at most one component with more than one vertex. It follows that B is an induced subgraph of $K_2 + (n-2)K_1$, as claimed. $\square$

By Definition 3.9 a graph S switches to a graph T if and only if $\overline{S}$ switches to $\overline{T}$, which leads to the following corollary.

Corollary 3.23. *For $n \geq 4$, the graph K_n switches to B if and only if B is an induced subgraph of $K_n - e$.*

Lemma 3.24. *For $n \geq 4$, the graph $K_2 + (n-2)K_1$ switches to B if and only if B is K_2 or cK_1, where $c \leq n-2$.*

Proof. Let H be a graph inducing $K_2+(n-2)K_1$, and let H' be a $\{K_2+(n-2)K_1\}$-free graph obtained from H after a single 2-switch. Since H' must contain at least one edge, $K_2 + (n-2)K_1$ switches to K_2. Furthermore, if U is the vertex set of some induced $K_2 + (n-2)K_1$ in H, then every 2-switch on H resulting in a $\{K_2 + (n-2)K_1\}$-free graph involves exactly two vertices of U; it follows that $H'[U]$ contains an independent set of size at least $n-2$, so $K_2+(n-2)K_1$ switches to cK_1 for each c at most $n-2$.

We now show that $K_2 + (n-2)K_1$ switches to no other graphs than the ones described above. Let H_1 be the graph K' shown in Figure 3.4, and let $H_1' = K_{2,n}$. Let H_2 be the graph shown on the left in Figure 3.5, consisting of $n-4$ isolated vertices and one nontrivial component on six vertices, and let $H_2' = K_4 + K_2 + (n-4)K_1$.

Suppose that $K_2 + (n-2)K_1$ switches to B. The graphs H_1 and H_2 induce $K_2 + (n-2)K_1$ while the graphs H_1' and H_2' do not, so B is induced in both H_1' and H_2'. Since H_1' is a complete bipartite graph (and hence triangle-free and $\{K_2 + K_1\}$-free), B must be these things as well. Since B is induced in H_2', it must also be a disjoint union of at most $n-2$ complete graphs. It follows that B is isomorphic to either K_2 or to cK_1 for some $c \leq n-2$. □

Corollary 3.25. *For $n \geq 4$, the graph $K_n - e$ switches to B if and only if B is $2K_1$ or K_c, where $c \leq n-2$.*

We now our results on switching to show that a large family of pairs are not degree-sequence-forcing.

Proposition 3.26. *For $n \geq 4$, the following are not degree-sequence-forcing pairs for any B on 3 or more vertices: $\{nK_1, B\}$, $\{K_n, B\}$, $\{K_2 + (n-2)K_1, B\}$, and $\{K_n - e, B\}$.*

Proof. Suppose that $\{nK_1, B\}$ is degree-sequence-forcing. By Proposition 3.10, nK_1 switches to B, so by Lemma 3.22 B must be an induced subgraph of $K_2 + (n-2)K_1$. If B has no edges, then it is induced in nK_1, and by Proposition 3.4 the set $\{B\}$ is degree-sequence-forcing. This contradicts Theorem 3.17, since B has at least 3 vertices; thus B has an edge and hence is of the form $n'K_1 + e$ for some n' at most n. However, Proposition 3.10 implies that B switches to nK_1, and by Lemma 3.24 we have that $n' \geq n+2$, a contradiction. Thus $\{nK_1, B\}$ is not degree-sequence-forcing.

Suppose that $\{K_2 + (n-2)K_1, B\}$ is degree-sequence-forcing. By Proposition 3.10, $K_2 + (n-2)K_1$ switches to B, and Lemma 3.24 implies that B has the form cK_1 for $c \leq n-2$, since B has at least three vertices. However, B must switch to $K_2 + (n-2)K_1$, and this contradicts Lemma 3.22, since $K_2 + (n-2)K_1$ has more vertices than B. Thus $\{K_2 + (n-2)K_1, B\}$ is not degree-sequence-forcing.

By Proposition 3.6, neither $\{K_n, B\}$ nor $\{K_n - e, B\}$ are degree-sequence-forcing for any B on at least three vertices. $\square$

We now finish the proof of Theorem 3.18

Proof of Theorem 3.18. By the results of this section it suffices to suppose that $\mathcal{F}$ is a degree-sequence-forcing set, where $\mathcal{F} = \{F_1, F_2\}$ and both F_1 and F_2 have at least four vertices. We must show that $\mathcal{F} = \{2K_2, C_4\}$. By Propositions 3.12 and 3.15 and Corollaries 3.14 and 3.16, $\mathcal{F}$ must contain an element from each of $\mathbb{K}$, $\mathbb{K}^c$, $\mathbb{S}$, and $\mathbb{S}^c$. It is easy to see that $\mathbb{K} \cap \mathbb{K}^c$ contains only complete or edgeless graphs, so by Proposition 3.26, one of F_1 and F_2 belongs to $\mathbb{K}$ while the other belongs to $\mathbb{K}^c$. Without loss of generality, assume that $F_1 \in \mathbb{K}$ and $F_2 \in \mathbb{K}^c$. We also have $\mathbb{S} \cap \mathbb{S}^c = \{K_1, K_2, 2K_1, K_2 + K_1, P_3\}$, so since F_1 and F_2 both have at least four vertices, one F_i belongs to $\mathbb{S}$ while the other belongs to $\mathbb{S}^c$.

Suppose that $F_1 \in \mathbb{S}^c$ and $F_2 \in \mathbb{S}$. The class $\mathbb{K} \cap \mathbb{S}^c$ consists of complete graphs and complete graphs plus an isolated vertex. By Proposition 3.26, we may write $F_1 = K_a + K_1$, where $a \geq 3$. Since $\mathbb{K}^c \cap \mathbb{S}$ consists of edgeless graphs and stars, Proposition 3.26 also implies that $F_2 = K_{1,b}$ for some integer $b \geq 3$. If $H = K_{2,b}$, and let H' be the graph K' from Figure 3.4 (where $n = b$), then (H, H') is an $\mathcal{F}$-breaking pair, a contradiction.

Suppose instead that $F_1 \in \mathbb{S}$ and $F_2 \in \mathbb{S}^c$. The class $\mathbb{K} \cap \mathbb{S}$ consists of disjoint unions of complete graphs with one or two vertices each, and the class $\mathbb{K}^c \cap \mathbb{S}^c$ consists of complete multipartite graphs where every partite set has size at most 2. Thus $F_1 = aK_2 + bK_1$ for nonnegative integers a and b; by Proposition 3.26, $a \geq 2$. Proposition 3.26 also implies that at least two of the partite sets in F_2 have size 2, so F_2 induces C_4. If H and H' are the graphs formed by taking the disjoint union of $(a-2)K_2 + bK_1$ with the co-4-pan and 4-pan graphs, respectively, then (H, H') is $\{F_1\}$-breaking. It follows that F_2 is induced in H', and since the only

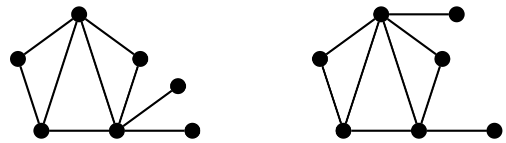


Figure 3.6: A $\{2K_2, C_4, K_{1,4}\}$-breaking pair.

complete multipartite induced subgraph of F_2 containing C_4 is C_4 itself, we have $F_2 = C_4$. By Proposition 3.21 we conclude that $\mathcal{F} = \{2K_2, C_4\}$. □

3.4 Non-minimal degree-sequence-forcing triples

In comparing Theorems 3.17 and 3.18, we notice that by appending any graph to a degree-sequence-forcing singleton set we obtain a degree-sequence-forcing pair, though there are degree-sequence-forcing pairs that do not contain a degree-sequence-forcing singleton. We define a degree-sequence-forcing set to be *minimal* if it contains no proper subset that is degree-sequence-forcing, and *non-minimal* otherwise.

For example, from Table 3.1 we observe that the set of forbidden subgraphs for the class of matroidal graphs is a non-minimal degree-sequence-forcing set, since it properly contains the set of forbidden subgraphs for the class of matrogenic graphs. Likewise, $\{2K_2, C_4, C_5\}$ and $\{2K_2, C_4, P_4\}$ are nonminimal degree-sequence-forcing sets, since they contain the (minimal) degree-sequence-forcing pair $\{2K_2, C_4\}$. In this section we study non-minimal degree-sequence-forcing sets, turning to the minimal degree-sequence-forcing sets in the next section.

We note that not every set of graphs that contains a degree-sequence-forcing subset is degree-sequence-forcing. For example, though the set $\{2K_2, C_4\}$ is degree-sequence-forcing, the set $\{2K_2, C_4, K_{1,4}\}$ is not: the graphs in Figure 3.6 constitute a $\{2K_2, C_4, K_{1,4}\}$-breaking pair. We characterize all non-minimal degree-sequence-forcing triples with the next theorem, whose proof will be the focus of

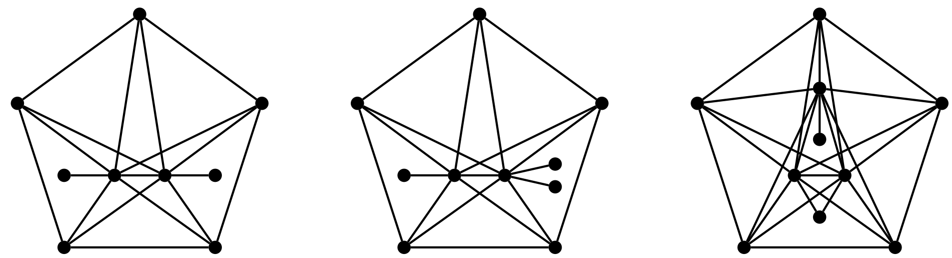


Figure 3.7: The graphs from Theorem 1(viii).

this section. The *chair* graph is the unique 5-vertex graph with degree sequence $(3, 2, 1, 1, 1)$; the *kite* graph is its complement.

Theorem 3.27. *A set $\mathcal{F}$ of 3 graphs is a non-minimal degree-sequence-forcing set if and only if one of the following conditions holds:*

(1) *$\mathcal{F}$ contains a proper degree-sequence-forcing subset other than $\{2K_2, C_4\}$;*

(2) *$\mathcal{F} = \{2K_2, C_4, F\}$, where F satisfies one of the following:*

   (i) *F induces $2K_2$ or C_4;*

   (ii) *$F \cong nK_1$ or $F \cong K_n$ for some $n \geq 1$;*

   (iii) *$F \cong C_5 + nK_1$ or $F \cong C_5 \vee K_n$ for some $n \geq 0$;*

   (iv) *$F \cong ((C_5 + nK_1) \vee K_1) + mK_1$ or $F \cong ((C_5 \vee K_n) + K_1) \vee K_m$ for some $m, n \geq 0$;*

   (v) *$F \cong K_2 + (n-2)K_1$ or $F \cong K_n - e$ for some $n \geq 2$;*

   (vi) *F or $\overline{F}$ is isomorphic to $((C_5 \vee K_1) + 2K_1) \vee K_1$;*

   (vii) *F has 4 or fewer vertices or is isomorphic to the chair or kite;*

   (viii) *F is isomorphic to one of the graphs in Figure 3.7;*

   (ix) *$F \cong K_{1,3} + K_1$ or $F \cong (K_3 + K_1) \vee K_1$.*

We give the proof in stages. In Section 3.4.1 we show that each of the triples from Theorem 3.27 is degree-sequence-forcing, and in Section 3.4.2 we show that there are no other non-minimal degree-sequence-forcing triples by studying an analogue of degree-sequence-forcing sets in the context of bipartite graphs.

3.4.1 Proof of sufficiency in Theorem 3.27

We begin with a few basic results.

Remark 3.28. *By Proposition 3.6, the set* $\{2K_2, C_4, F\}$ *is degree-sequence-forcing if and only if* $\{2K_2, C_4, \overline{F}\}$ *is, since* $2K_2$ *and* C_4 *are complements of each other.*

Remark 3.29. *Let* $\mathcal{G}$ *be a family of graphs. If* (H, H') *is a* $\{2K_2, C_4\} \cup \mathcal{G}$*-breaking pair, then* (H, H') *is a* $\mathcal{G}$*-breaking pair, and* H *and* H' *are both* $\{2K_2, C_4\}$*-free.*

Non-minimal degree-sequence-forcing triples are formed by appending suitable graphs to the degree-sequence-forcing sets from Theorems 3.17 and 3.18. For degree-sequence-forcing proper sets other than $\{2K_2, C_4\}$, we may append any graphs we wish, as the following remark shows.

Remark 3.30. *If a set* $\mathcal{F}$ *contains a degree-sequence-forcing singleton or pair other than* $\{2K_2, C_4\}$*, then the* $\mathcal{F}$*-free graphs are unigraphs by Proposition 3.19. By Remark 3.7,* $\mathcal{F}$ *is a degree-sequence-forcing set.*

This proves that the sets listed in item 1 of Theorem 3.27 are degree-sequence-forcing. We now examine the triple $\mathcal{F} = \{2K_2, C_4, F\}$. By Proposition 3.4, $\mathcal{F}$ is degree-sequence-forcing if F induces $\{2K_2, C_4\}$, as stated in item 2(i).

We henceforth assume that F is $\{2K_2, C_4\}$-free. The next several definitions and results provide a framework for discussion of the structure of $\{2K_2, C_4\}$-free graphs and the 2-switches possible on them. We begin with a structural characterization of $\{2K_2, C_4\}$-free graphs due to Blázsik et al. [7].

Theorem 3.31. *A graph* G *is* $\{2K_2, C_4\}$*-free if and only if there exists a partition* V_1, V_2, V_3 *of* $V(G)$ *such that*

(i) *V_1 is an independent set,*

(ii) *V_2 is a clique,*

(iii) *$V_3 = \emptyset$ or $G[V_3] \cong C_5$,*

(iv) *every possible edge exists between V_2 and V_3, and*

(v) *no edge in G has one endpoint in V_1 and the other endpoint in V_3.*

Given a $\{2K_2, C_4\}$-free graph G, we call the triple (V_1, V_2, V_3) a *pseudo-splitting partition* of $V(G)$ if V_1, V_2, V_3 satisfy the conditions of the partition set forth in Theorem 3.31. The name is suggested by [36], in which $\{2K_2, C_4\}$-free graphs are called *pseudo-split* graphs. Similarly, for a split graph G, define a *splitting partition* of $V(G)$ to be an ordered partition (V_1, V_2) of $V(G)$ into an independent set and a clique, respectively.

Note that there is at most one induced C_5 in any $\{2K_2, C_4\}$-free graph. Given a $\{2K_2, C_4\}$-free graph C, define the *split part* G^s of G to be the induced subgraph resulting from deleting the vertices of the induced C_5 from G if such a 5-cycle exists, and letting $G^s = G$ otherwise.

The following is an easy consequence of Theorem 3.31.

Corollary 3.32. *Let H be an arbitrary $\{2K_2, C_4\}$-free graph, and let (W_1, W_2, W_3) be a pseudo-splitting partition of $V(H)$. Any induced P_4 in H either lies in $H[W_3]$ or has its endpoints in W_1 and its midpoints in W_2.*

Proposition 3.33. *Let H be an arbitrary $\{2K_2, C_4\}$-free graph with pseudo-splitting partition (W_1, W_2, W_3). Let H' be a graph obtained via a 2-switch on H, with $H' \ncong H$. The following statements all hold.*

(i) *The 2-switch changing H into H' is performed on a set of vertices in $W_1 \cup W_2$ on which a P_4 is induced in both H and H'.*

(ii) *The triple* (W_1, W_2, W_3) *is a pseudo-splitting partition of* H'.

(iii) *For any* $u \in W_1$ *and* $v \in W_2$*, we have* $|N_H(u) \cap W_2| = |N_{H'}(u) \cap W_2|$ *and* $|N_H(v) \cap W_1| = |N_{H'}(v) \cap W_1|$.

Proof. (i) The four vertices on which the 2-switch is performed must induce $2K_2$, C_4, or P_4; since the first two graphs are forbidden in H, the 2-switch must have occurred on an induced P_4. Any 2-switch on an induced P_4 leaves an induced P_4 on the four vertices involved. By Corollary 3.32 the P_4 must either be located entirely within W_3 or within $W_1 \cup W_2$. From Theorem 3.31, any 2-switch on a P_4 contained in $G[W_3]$ will not change the isomorphism class of the graph, since every 2-switch on a copy of C_5 produces another copy of C_5, every vertex of W_2 dominates the induced C_5 in both H and H', and every vertex of W_1 is nonadjacent to every vertex of the induced C_5. Hence the isomorphism-class-changing 2-switch must occur on the vertex set of an induced P_4 in $G[W_1 \cup W_2]$.

(ii) Let $abcd$ be the induced P_4 on which the 2-switch changing H into H' occurred. By (i) and Corollary 3.32, $a, d \in W_1$ and $b, c \in W_2$. Note that in the 2-switch the edges deleted are ab, cd and the edges added are ad, bc. Thus after the 2-switch no edge exists between vertices in W_1, no non-edge exists in W_2, and all the other requirements for (W_1, W_2, W_3) to be a pseudo-splitting partition hold.

(iii) This is clear upon considering the edges deleted and added as part of the 2-switch in the proof of (ii). $\square$

Lemma 3.34. *Let* H *be a* $\{2K_2, C_4\}$*-free graph with pseudo-splitting partition* (W_1, W_2, W_3)*, and let* H' *be a graph obtained by performing a 2-switch on* H*. If* G *is an induced subgraph of* H *that is not induced in* H'*, then* $|V(G) \cap W_2| \geq 2$.

Proof. Let H, H', and G be as described in the hypothesis, and let $V_2 = V(F) \cap W_2$. Since $H[W_1 \cup W_3] \cong H'[W_1 \cup W_3]$ and H' is F-free, we have $|V_2| \geq 0$.

Suppose that $|V_2| = 1$, and let $V_2 = \{v\}$. Let p_1 and p_2 denote respectively the number of vertices in $V(F) \cap W_1$ to which v is and is not adjacent to. By Proposition 3.33, after the 2-switch that creates H' there are still at least p_1 vertices in W_1 to which v is adjacent, and at least p_2 vertices in W_1 to which v is not adjacent. These $p_1 + p_2$ vertices, together with v and $V(F) \cap W_3$, clearly induce F in H', a contradiction; thus $|V_2| \geq 2$, as claimed. $\square$

We now show that the sets described in items 2(ii), 2(iii), and 2(iv) of Theorem 3.27 are degree-sequence-forcing.

Corollary 3.35. *The set $\mathcal{F} = \{2K_2, C_4, F\}$ is degree-sequence-forcing whenever F is one of the following:*

(i) nK_1, $n \geq 1$;

(ii) K_n, $n \geq 1$;

(iii) $C_5 + nK_1$, $n \geq 0$;

(iv) $C_5 \vee K_n$, $n \geq 0$;

(v) $((C_5 + nK_1) \vee K_1) + mK_1$, $m, n \geq 0$;

(vi) $((C_5 \vee K_n) + K_1) \vee K_m$, $m, n \geq 0$.

Proof. If F is any of the graphs described in (i), (iii), or (v), then every pseudo-splitting partition (V_1, V_2, V_3) of $V(F)$ has $|V_2| \leq 1$. By Lemma 3.34 and Remark 3.29, there exist no $\{2K_2, C_4, F\}$-breaking pairs, so $\{2K_2, C_4, F\}$ is degree-sequence-forcing. The cases (ii), (iv), and (vi) follow from the cases (i), (iii) and (v) by Proposition 3.6. $\square$

We now consider sets of the form listed in item 2(v) of Theorem 3.27.

Proposition 3.36. *The triples* $\mathcal{F} = \{2K_2, C_4, K_2+(n-2)K_1\}$ *and* $\mathcal{G} = \{2K_2, C_4, K_n - e\}$ *are degree-sequence-forcing for all* $n \geq 2$.

Proof. By Proposition 3.6, it suffices to show that the triple $\mathcal{F}$ is degree-sequence-forcing.

Corollary 3.35 handles the case when $n = 2$, so we assume that $n \geq 3$. Suppose that $\mathcal{F}$ is not degree-sequence-forcing, and let (H, H') be an $\mathcal{F}$-breaking pair. By Remark 3.29, (H, H') is an $\{K_2+(n-2)K_1\}$-breaking pair where both H and H' are $\{2K_2, C_4\}$-free. Let $V = \{a, b, i_3, i_4, \dots, i_n\}$ be the vertex set of an induced copy of $K_2 + (n-2)K_1$ in H, with $ab \in E(H)$. By Proposition 3.33, we may fix a pseudo-splitting partition (W_1, W_2, W_3) of both $V(H)$ and $V(H')$. From Lemma 3.34 it follows that $|V \cap W_2| = 2$, so $V \cap W_2 = \{a, b\}$, and $V - \{a, b\} \subseteq W_1$. Graph H' is $\{K_2+(n-2)K_1\}$-free, so the 2-switch transforming H into H' must add an edge between either a or b and i_k for some $k \in \{3, \dots, n\}$; without loss of generality assume the edge ai_3 is added. The 2-switch must be $\{ax, i_3y\} \rightrightarrows \{ai_3, xy\}$ for some $x \in W_1$ and some $y \in W_2$. However, then $H'[\{a, x, i_3, \dots, i_n\}] \cong K_2 + (n-2)K_1$, a contradiction. Thus $\mathcal{F}$ is degree-sequence-forcing. □

The next result addresses the sets described in item 2(vi) of Theorem 3.27.

Proposition 3.37. *The sets* $\{2K_2, C_4, F\}$ *and* $\{2K_2, C_4, \overline{F}\}$ *are degree-sequence-forcing, where* $F \cong ((C_5 \vee K_1) + 2K_1) \vee K_1$.

Proof. Suppose that (H, H') is a $\{2K_2, C_4, F\}$-breaking pair, where $F \cong ((C_5 \vee K_1) + 2K_1) \vee K_1$. By Remark 3.29 and Proposition 3.33, we may assume that H induces F and that H and H' have the same vertex set and a common pseudo-splitting partition (W_1, W_2, W_3). Fix a copy of F in H. Note that there is a unique pseudo-splitting partition of F, and it must be $(W_1 \cap V(F), W_2 \cap V(F), W_3 \cap V(F))$. Within the induced copy of F, let c and ℓ_1 be the vertices having degrees 8 and 6 in H, respectively, and let ℓ_2 and ℓ_3 be the pendant vertices. By Proposition 3.11

and Remark 3.29, we may assume that (H, H') is an F-breaking pair, and that H contains at most 2 vertices not contained in the copy of F. In order for there to be an isomorphism-class-changing 2-switch on H, there must be an induced copy of P_4 on $H[W_1 \cup W_2]$ that includes a vertex from each of $W_1 \cap V(F)$ and $W_2 \cap V(F)$. There then exists a vertex $y \in W_1 - V(F)$ such that y has a neighbor other than c in W_2 and y does not dominate W_2. If $|W_2| = 2$, then y is adjacent to ℓ_1 but not to c; but then any 2-switch involving y has the form $\{\ell_1 y, vc\} \rightrightarrows \{\ell_1 v, yc\}$ for some $v \in W_1 - \{y\}$, and $H'[\{y, w\} \cup W_2 \cup W_3]$ is a copy of F, where $w \in \{\ell_2, \ell_3\} - v$. Thus H contains a vertex $x \in W_2$ that does not belong to F, and $V(H) = V(F) \cup \{x, y\}$.

If $N_H(y) = \{x\}$, then the 2-switch changing H into H' must be $\{xy, vc\} \rightrightarrows \{xv, yc\}$, where $v \in \{\ell_2, \ell_3\}$; since $xv \notin E(H)$, the 2-switch in effect merely switches the roles of y and v without changing the isomorphism class of H. A similar contradiction arises if $N_H(y) = \{\ell_1\}$. The neighborhood of y in H cannot be $\{x, c\}$ or $\{\ell_1, c\}$ or $\{x, c, \ell_1\}$, for no 2-switch would then be possible on H. Thus $N_H(y) = \{\ell_1, x\}$, and the 2-switch changing H into H' then has the form $\{vy, wc\} \rightrightarrows \{vw, yc\}$, where $v \in \{\ell_1, x\}$ and $w \in \{\ell_2, \ell_3\}$. If $v = \ell_1$ then $\ell_1 y \notin E(H')$, and $H'[W_3 \cup \{c, \ell_1, y, u\}] \cong F$, where $u \in \{\ell_2, \ell_3\} - w$. If $v = x$ then since $xy \notin E(H')$ and H' cannot induce F on $W_3 \cup \{c, x, y, u\}$, where $u \in \{\ell_2, \ell_3\} - w$, we must have $xu \in E(H')$; but then $H'[W_3 \cup \{x, \ell_1, \ell_2, \ell_3\}] \cong F$, a contradiction.

We conclude that no $\{2K_2, C_4, F\}$-breaking pair exists, so this set is degree-sequence-forcing. By Proposition 3.6, it follows that $\{2K_2, C_4, \overline{F}\}$ is also degree-sequence-forcing. $\square$

We now prove that the sets described in item 2(vii) of Theorem 3.27 are degree-sequence-forcing, beginning with a few technical results.

Lemma 3.38. *If G is a $\{2K_2, C_4\}$-free graph and $abcd$ is an induced copy of P_4 such that every vertex not in $\{a, b, c, d\}$ is adjacent to exactly 0 or 2 vertices in*

$\{a, d\}$, *then the graph* G' *formed by performing the* 2*-switch* $\{ab, cd\} \rightrightarrows \{ac, bd\}$ *is isomorphic to* G.

Proof. It is easy to verify that the bijection from $V(G)$ to $V(G')$ that maps a and d to each other and fixes every other element (recall that $V(G) = V(G')$ is an isomorphism. $\square$

Lemma 3.39. *If* $abcd$ *is an induced copy of* P_4 *in a* $\{2K_2, C_4, \text{kite}\}$*-free graph* G, *then every vertex of* $V(G) - \{a, b, c, d\}$ *is adjacent to exactly* 0 *or* 2 *of* $\{a, d\}$.

Proof. Let (V_1, V_2, V_3) be a pseudo-splitting partition of G. By Corollary 3.32, either the path $\langle a, b, c, d\rangle$ is contained within $G[V_3]$, in which case the claim is clearly true by Theorem 3.31, or $a, d \in V_1$ and $b, c \in V_2$. Assume that the latter holds, and suppose that $u \neq b$ and $ua \in E(G)$. It follows that $u \in V_2$, so $ub, uc \in E(G)$. Since G does not induce the kite on $\{a, b, c, d, u\}$, we must have $ud \in E(G)$. Similar arguments show that any vertex other than c adjacent to d must also be adjacent to a, and the result follows. $\square$

Proposition 3.40. *If* $\mathcal{F}$ *is* $\{2K_2, C_4, \text{kite}\}$ *or* $\{2K_2, C_4, \text{chair}\}$, *then the* $\mathcal{F}$*-free graphs are unigraphs.*

Proof. Any 2-switch on a $\{2K_2, C_4, \text{kite}\}$-free graph G must be performed on an induced copy of P_4. Lemmas 3.38 and 3.39 imply that the graph resulting from such a 2-switch is isomorphic to G, so by Theorem 3.8 G is a unigraph. Since every $\{2K_2, C_4, \text{chair}\}$-free graph H is the complement of a $\{2K_2, C_4, \text{kite}\}$-free graph, H is also a unigraph. $\square$

Corollary 3.41. *The triple* $\mathcal{F} = \{2K_2, C_4, F\}$ *is degree-sequence-forcing if* F *is the kite or chair graph, or any graph on* 4 *or fewer vertices.*

Proof. If $F \in \{2K_2, C_4\}$ then $\mathcal{F} = \{2K_2, C_4\}$ which by Theorem 3.18 is degree-sequence-forcing. If $F \in \{4K_1, K_4\}$, then by Corollary 3.35 $\mathcal{F}$ is degree-sequence-forcing. If F is any other graph on 4 or fewer vertices, or if F is the chair or kite graph, then F is induced in either the chair or the kite graph. The $\{2K_2, C_4, F\}$-free graphs are thus $\{2K_2, C_4, \text{chair}\}$-free or $\{2K_2, C_4, \text{kite}\}$-free and hence are unigraphs by Proposition 3.40, so $\{2K_2, C_4, F\}$ is degree-sequence-forcing by Remark 3.7. $\square$

In the next few results we show that the graphs in item 2(viii) of Theorem 3.27 (those illustrated in Figure 3.7) complete a degree-sequence-forcing triple with $\{2K_2, C_4\}$. We begin by establishing some facts about the triple $\mathcal{F}$ where F induces C_5.

Proposition 3.42. *Let G and H be C_5-inducing graphs that are $\{2K_2, C_4\}$-free. If the vertex set of the split part G^s of G has a unique partition into a clique and an independent set, then H is G-free if and only if H^s is G^s-free.*

Proof. Suppose first that H^s induces G^s, and assume that $V(G^s) \subseteq V(H^s)$. Let (W_1, W_2, W_3) be the pseudo-splitting partition of H. The ordered partition (W_1, W_2) is a splitting partition of $V(H^s)$, and if (V_1, V_2) is a splitting partition of $V(G^s)$, then the uniqueness of the latter partition forces $V_1 \subseteq W_1$ and $V_2 \subseteq W_2$. Now in H there is an induced copy of C_5 on V_3 in which every vertex dominates W_2. Since G is constructed by making each vertex of a copy of C_5 adjacent to each vertex in V_2 and not adjacent to any vertex in V_1, it is clear that G is induced in H.

For the converse, suppose that H induces G, and assume $V(G) \subseteq V(H)$. Let (W_1, W_2, W_3) be a pseudo-splitting partition of H, and let (V_1, V_2, V_3) be a pseudo-splitting partition of G. Now H induces a single copy of C_5, as does G, so $V_3 = W_3$. Then $V_1 \cup V_2 \subseteq W_1 \cup W_2$, and it is clear that G^s is induced in H^s. $\square$

Proposition 3.43. *Suppose that G is a $\{2K_2, C_4\}$-free graph such that G induces C_5, and $V(G^s)$ has a unique partition into a clique and an independent set.*

If $\{2K_2, C_4, G^s\}$ is degree-sequence-forcing, then $\{2K_2, C_4, G\}$ is degree-sequence-forcing as well.

Proof. Suppose that $\mathcal{G} = \{2K_2, C_4, G\}$ is not degree-sequence-forcing, and let (H_1, H_2) be a $\mathcal{G}$-breaking pair. Proposition 3.42 implies that H_1^s induces G^s, and H_2^s is G^s-free; thus (H_1^s, H_2^s) is a G^s-breaking pair and hence a $\{2K_2, C_4, G^s\}$-breaking pair. We conclude that $\{2K_2, C_4, G^s\}$ is not degree-sequence-forcing. $\square$

Corollary 3.44. *Let F be the unique C_5-inducing $\{2K_2, C_4\}$-free graph such that $F^s \cong$ chair. Let G be the unique C_5-inducing $\{2K_2, C_4\}$-free graph such that $G^s \cong P_4$. The sets $\{2K_2, C_4, F\}$, $\{2K_2, C_4, \overline{F}\}$, and $\{2K_2, C_4, G\}$ are degree-sequence-forcing.*

We conclude our proof of the sufficiency of the conditions listed in Theorem 3.27 by addressing the case listed in item 2(ix) of Theorem 3.27.

Proposition 3.45. *The sets $\{2K_2, C_4, K_{1,3}+K_1\}$ and $\{2K_2, C_4, (K_3+K_1) \vee K_1\}$ are degree-sequence-forcing.*

Proof. Let $\mathcal{F} = \{2K_2, C_4, K_{1,3} + K_1\}$. If $\mathcal{F}$ is not degree-sequence-forcing, then there exists a $\{K_{1,3}+K_1\}$-breaking pair (H_1, H_2) of $\{2K_2, C_4\}$-free graphs, and we may assume that H_2 is obtained by performing a single 2-switch on H_1, so that the two have the same vertex set. Let (W_1, W_2, W_3) be a pseudo-splitting partition of $V(H_1)$. It follows from Lemma 3.34 that any induced subgraph isomorphic to $K_{1,3}+K_1$ contains exactly two vertices from W_2. Fix an induced subgraph G of H_1 isomorphic to $K_{1,3} + K_1$. It follows from Theorem 3.31 that G has three vertices s, ℓ_2, and ℓ_3 in W_1 and two vertices c and ℓ_1 in W_2, with c and s the vertices of degrees 3 and 0, respectively, in G. Since H_2 induces no copy of $K_{1,3} + K_1$ on $V(G)$, it follows from Proposition 3.33 that the 2-switch changing H_1 into H_2 must either add an edge between ℓ_1 and one of ℓ_2 or ℓ_3, add the edge $\ell_1 s$ to G, add the edge cs to G, or delete an edge joining c and one of ℓ_2 or ℓ_3. We consider each of these possibilities.

If the 2-switch adds an edge between ℓ_1 and either ℓ_2 or ℓ_3, without loss of generality we may assume that the 2-switch has the form $\{\ell_1 y, \ell_2 x\} \rightrightarrows \{\ell_1\ell_2, yx\}$ for some vertex $x \in W_2$ and some $y \in W_1$ We make no initial assumption that x or y is distinct from a vertex in $V(G)$ (other than the vertices ℓ_1 and x involved in the 2-switch); however, since $\ell_1 y$ is an edge in H_1, we deduce that $y \notin \{\ell_2, \ell_3\}$. If cy were an edge of H_1, then H_1 would induce a copy of $K_{1,3}+K_1$ on $\{c, \ell_2, \ell_3, s, y\}$ having only one vertex in W_2, a contradiction; thus $cy \notin E(H_1)$. Now in H_2 we have edges cx, $c\ell_2$, $c\ell_3$ and non-edges $x\ell_2$, $\ell_2\ell_3$, $s\ell_2$, $s\ell_3$; since H_2 is $\{K_{1,3}+K_1\}$-free, H_2 must contain either $x\ell_3$ or xs as an edge. If H_2 had both of these edges then it would induce $K_{1,3}+K_1$ on $\{\ell_2, \ell_3, s, x, y\}$, a contradiction, so exactly one of $x\ell_3$ or xs is and edge of G. In either case H_2 induces $K_{1,3}+K_1$ on $\{\ell_1, \ell_3, s, x, y\}$, a contradiction.

If instead the 2-switch producing H_2 adds the edge $\ell_1 s$, then the 2-switch has the form $\{\ell_1 y, sx\} \rightrightarrows \{\ell_1 s, yx\}$ for some $x \in W_2$ and $y \in W_1$. Since x is adjacent to s, we have $x \notin \{c, \ell_1\}$, and since y is adjacent to ℓ_1, we have $\{y \notin \ell_2, \ell_3, s\}$. Since $\{c, \ell_2, \ell_3, s, y\}$ cannot induce $K_{1,3}+K_1$ in H_1 (only one of these vertices belongs to W_2), we have $cy \notin E(H_1)$. In H_2 we have y adjacent to neither of c or ℓ_1, so H_2 induces $K_{1,3}+K_1$ on $\{c, \ell_1, \ell_2, \ell_3, y\}$, a contradiction.

If the 2-switch adds instead the edge cs to G, then the 2-switch performed has the form $\{cy, sx\} \rightrightarrows \{cs, xy\}$ for some $x \in W_2$ and $y \in W_1$. Since x is adjacent to s in H_1, we have $x \neq \ell_1$. However, since H_1 cannot induce $K_{1,3}+K_1$ on $\{c, \ell_2, \ell_3, s, y\}$, we must have $y \in \{\ell_2, \ell_3\}$. Without loss of generality we assume that $y = \ell_2$, so that the 2-switch performed is $\{c\ell_2, sx\} \rightrightarrows \{cs, x\ell_2\}$. The subgraph of H_2 induced on $\{c, \ell_1, \ell_2, \ell_3, s\}$ is then isomorphic to $K_{1,3}+K_1$, a contradiction.

Finally, if the 2-switch changing H_1 into H_2 deletes an edge joining c to one of

ℓ_2 or ℓ_3, then we may assume without loss of generality that the 2-switch has the form $\{c\ell_2, yx\} \rightrightarrows \{cy, \ell_2 x\}$ for some $x \in W_2$ and $y \in W_1$. Since xy is an edge in H_1, we cannot have both $x = \ell_1$ and $y = s$. If $y \neq s$, then H_2 induces $K_{1,3} + K_1$ on $\{c, \ell_3, s, x, y\}$ unless x is adjacent to ℓ_3 or s. Since H_2 is $\{K_{1,3} + K_1\}$-free, x cannot be adjacent to both vertices, and if x is adjacent to either one, then H_2 induces $K_{1,3} + K_1$ on $\{\ell_1, \ell_2, \ell_3, s, x\}$, a contradiction. Thus $y = s$ and $x \neq \ell_1$; in this case H_2 induces $K_{1,3} + K_1$ on $\{c, \ell_1, \ell_2, \ell_3, s\}$, a contradiction.

Since every case yields a contradiction, we conclude that no $\mathcal{F}$-breaking pair exists. Thus $\{2K_2, C_4, K_{1,3} + K_1\}$ is degree-sequence-forcing, and by Proposition 3.6 the set $\{2K_2, C_4, (K_3 + K_1) \vee K_1\}$ is degree-sequence-forcing as well. □

3.4.2 Proof of necessity in Theorem 3.27

In this subsection we show that the only non-minimal degree-sequence-forcing triples are the ones presented in Theorem 3.27. In the previous section, we used an observation on order-preserving graph parameters (Remark 3.13) to show that every degree-sequence-forcing set contains graphs of certain types (namely, from the classes $\mathbb{K}$, $\mathbb{K}^c$, $\mathbb{S}$, and $\mathbb{S}^c$). These results were useful in showing that no other degree-sequence-forcing pairs existed than those listed in Theorem 3.18.

In this subsection we again seek to show that all but certain specified sets are not degree-sequence-forcing. However, our assumption that the degree-sequence-forcing triples are non-minimal renders Remark 3.13 and the results that accompany it useless, because by assumption the sets we consider contain a degree-sequence-forcing set as a subset, and hence contain elements from $\mathbb{K}$, $\mathbb{S}$, and any other graph class determined by values of an order-preserving parameter.

Instead, we use the structure of $\{2K_2, C_4\}$-free graphs to formulate a notion of a degree-sequence-forcing sets in the context of bipartite graphs. We then use our results to complete the proof of Theorem 3.27. We begin with several definitions.

A *bipartitioned graph* is a triple (G, V_1, V_2) where G is a bipartite graph with partite sets V_1 and V_2. We use $G(V_1, V_2)$ to denote the bipartitioned graph and refer to G as the underlying graph. We define two bipartitioned graphs $G(V_1, V_2)$ and $G'(V_1', V_2')$ to be *isomorphic* if there exists a graph isomorphism $\phi\colon V(G) \to V(G')$ such that $\phi(V_1) = V_1'$ (and hence $\phi(V_2) = V_2'$).

We define the *bicomplement* $\overline{G(V_1, V_2)}$ of a bipartitioned graph $G(V_1, V_2)$ to be the bipartitioned graph $H(V_2, V_1)$ such that $E(H) = \{uv \ : \ u \in V_2, v \in V_1, uv \notin E(G)\}$. Note that in the bicomplement the roles of V_1 and V_2 are interchanged.

Given a split graph G and a splitting partition (V_1, V_2) of $V(G)$, we define the *associated bipartitioned graph* to be $G^b(V_1, V_2)$, where G^b is formed by deleting all edges with both endpoints in V_2. Note that an arbitrary split graph may have more than one partition into an independent set and a clique, and hence more than one associated bipartitioned graph. If H is a pseudo-split graph that induces C_5 and has pseudo-splitting partition (V_1, V_2, V_3), then the associated bipartitioned graph is defined to be $H^b(V_1, V_2)$, where H^b is formed by deleting V_3 from H and removing all edges with both endpoints in V_2; equivalently, $H^b = (H^s)^b$. A pseudo-split graph that induces C_5 has exactly one bipartitioned graph associated with it.

We say a bipartitioned graph $H(W_1, W_2)$ is an *induced subgraph* of $G(V_1, V_2)$ if $W_i \subseteq V_i$ for $i \in \{1, 2\}$ and $H = G[W_1 \cup W_2]$. We will often be more interested in isomorphism classes of bipartitioned graphs than with specific graphs themselves; for that reason, we say that $G(V_1, V_2)$ is $F(X_1, X_2)$*-free* if there is no induced subgraph of $G(V_1, V_2)$ isomorphic to $F(X_1, X_2)$, and we say that $G(V_1, V_2)$ *induces* $F(X_1, X_2)$ if there exists an induced subgraph of $G(V_1, V_2)$ isomorphic to $F(X_1, X_2)$.

We define the *degree sequence* of a bipartitioned graph $G(V_1, V_2)$ to be the

Figure 3.8: The chair graph and its associated bipartitioned graph.

Figure 3.9: A bipartitioned 2-switch and a non-bipartitioned 2-switch.

ordered pair $(d; d')$, where d and d' are lists of the degrees in G of the vertices in V_1 and V_2, respectively, written in nonincreasing order. If $G(V_1, V_2)$ has degree sequence $(d; d')$, then we say that $G(V_1, V_2)$ is a *realization of* $(d; d')$.

Example 3.46. The chair graph G is shown on the left in Figure 3.8. Its unique associated bipartitioned graph $G^b(V_1, V_2)$ is shown on the right, with vertices in V_1 on the bottom row and vertices in V_2 on the top row. The degree sequence of $G^b(V_1, V_2)$ is $(1, 1, 1; 2, 1)$.

We note that nonisomorphic bipartitioned graphs may have a common degree sequence. We define a set $\mathcal{F} = \{F_1(V_1^1, V_2^1), \ldots, F_k(V_1^k, V_2^k)\}$ of bipartitioned graphs to be *degree-sequence-forcing* if whenever a bipartitioned graph $G(W_1, W_2)$ with degree sequence $(d; d')$ induces no element of $\mathcal{F}$, no other realization of $(d; d')$ induces an element of $\mathcal{F}$.

In examining degree-sequence-forcing sets of bipartitioned graphs, we begin with the following useful note:

Remark 3.47. *Given a set $\mathcal{F}$ of bipartitioned graphs, let $\mathcal{F}^c$ denote the collection of bicomplements of elements of $\mathcal{F}$. The set $\mathcal{F}$ is degree-sequence-forcing if and only if $\mathcal{F}^c$ is degree-sequence-forcing.*

We define a *bipartitioned 2-switch* on $G(V_1, V_2)$ as the deletion of two edges uv, xy of G and the addition of edges uy, xv not already belonging to G, where we require that $u, x \in V_1$ and $v, y \in V_2$, as shown on the left in Figure 3.9. As before, we denote this 2-switch by $\{uv, yx\} \rightrightarrows \{uy, vx\}$. A bipartitioned 2-switch is a 2-switch on the underlying graph. However, the definition of a bipartitioned 2-switch is more restrictive; after the 2-switch $\{uv, yx\} \rightrightarrows \{uy, vx\}$ on $G(V_1, V_2)$, the sets V_1, V_2 still partition $V(G)$ into two independent sets. This need not be the case for an arbitrary 2-switch on a bipartite graph, as shown on the right in Figure 3.9, where the bottom and top rows of vertices contain subsets of V_1 and V_2, respectively.

As with general 2-switches, a bipartitioned 2-switch does not change the degree of any vertex in the bipartitioned graph. We arrive at an analogue of Theorem 3.8:

Proposition 3.48. *Bipartitioned graphs $G(W_1, W_2)$ and $H(W_1, W_2)$ on the same vertex set satisfy $d_G(v) = d_H(v)$ for every vertex $v \in W_1 \cup W_2$ if and only if H can be obtained by performing a sequence of bipartitioned 2-switches on G.*

Proof. Let $G(W_1, W_2)$ and $H(W_1, W_2)$ be bipartitioned graphs as described in the hypothesis. We proceed by induction on $|W_1|$. When $|W_1| = 1$ there is nothing to prove; suppose that $|W_1| > 1$. Let u be a vertex of maximum degree Δ among vertices in W_1, and let $v_1, \ldots, v_\Delta$ be a set of vertices in W_2 with the Δ highest degrees among vertices in W_2. We show that by means of bipartitioned 2-switches we can arrive at a bipartitioned graph where $N(u) = \{v_1, \ldots, v_\Delta\}$. Suppose that $uv_i \notin E(G)$ for some $i \in \{1, \ldots, \Delta\}$. The vertex u has a neighbor w in $W_2 - \{v_1, v_2, \ldots, v_\Delta\}$. Since v_i has degree at least as large as that of w, the vertex v_i has a neighbor x in $W_1 - N(w)$. We may perform the 2-switch $\{uw, v_ix\} \rightrightarrows \{uv_i, wx\}$ and obtain a graph where $|N(u) \cap \{v_1, \ldots, v_\Delta\}|$ is larger than it previously was. Repeating this procedure as necessary, we arrive at a graph G^* where $N(u) = \{v_1, \ldots, v_\Delta\}$. We may also perform a sequence of 2-switches on $H(W_1, W_2)$ to form a graph $H^*(W_1, W_2)$ such that $N(u) = \{v_1, \ldots, v_\Delta\}$. The bipartitioned graphs $G^*(W_1, W_2) - u$ and $H^*(W_1, W_2) - u$ agree on the degrees of all vertices, and by the inductive hypothesis there exists a finite sequence of bipartitioned 2-switches that changes $G^*(W_1, W_2) - u$ into $H^*(W_1, W_2) - u$. None of these bipartitioned 2-switches involve the vertex u, so the bipartitioned 2-switches that change $G(W_1, W_2)$ into $G^*(W_1, W_2)$, followed by the same bipartitioned 2-switches that change $G^*(W_1, W_2) - u$ into $H^*(W_1, W_2) - u$, followed by the bipartitioned 2-switches that change $H^*(W_1, W_2)$ into $H(W_1, W_2)$, give a sequence of bipartitioned 2-switches that change $G(W_1, W_2)$ into $H(W_1, W_2)$. The result follows by induction. $\square$

We are now in a position to show the relationship between degree-sequence-forcing sets of graphs and degree-sequence-forcing sets of bipartitioned graphs.

Theorem 3.49. *Let $\mathcal{F}$ be a collection of $\{2K_2, C_4\}$-free graphs that either all induce C_5 or are all C_5-free. Let $\mathcal{G}$ be the set of all bipartitioned graphs associated with elements of $\mathcal{F}$. The set $\{2K_2, C_4\} \cup \mathcal{F}$ is a degree-sequence-forcing set of graphs if and only if $\mathcal{G}$ is a degree-sequence-forcing set of bipartitioned graphs.*

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be as defined above. Suppose first that $\{2K_2, C_4\} \cup \mathcal{F}$ is degree-sequence-forcing. Let $H(W_1, W_2)$ be a bipartitioned graph inducing $G(V_1, V_2)$, where $G(V_1, V_2)$ is an element of $\mathcal{G}$. By definition, $V_1 \subseteq W_1$ and $V_2 \subseteq W_2$. Let $H'(W_1, W_2)$ be any other realization of the degree sequence of $H(W_1, W_2)$, and let J_1 and J_2 be pseudo-split graphs for which H and H' are associated bipartitioned graphs, respectively, such that J_1 and J_2 induce C_5 if and only if the graphs in $\mathcal{F}$ do. It is clear that J_1 and J_2 have the same degree sequence, and that J_1 induces some element of $\mathcal{F}$. Since $\{2K_2, C_4\} \cup \mathcal{F}$ is degree-sequence-forcing, it follows that J_2 also induces some element of $\mathcal{F}$; thus $H'(W_1, W_2)$ induces some element of $\mathcal{G}$. We conclude that $\mathcal{G}$ is degree-sequence-forcing.

Conversely, let $\mathcal{G}$ be degree-sequence-forcing, and suppose that $\{2K_2, C_4\} \cup \mathcal{F}$ is not degree-sequence-forcing. By Remark 3.29 there exists a $\{2K_2, C_4\} \cup \mathcal{F}$-breaking pair (H_1, H_2) of $\{2K_2, C_4\}$-free graphs. There exists a sequence of 2-switches that transforms H_1 into H_2; by Proposition 3.33 there exists a partition W_1, W_2, W_3 of $V(H_1) = V(H_2)$ such that in both H_1 and H_2 the set W_1 is an independent set, W_2 is a clique, and W_3 is either empty or the vertex set of an induced C_5. Consider the bipartitioned graphs $H_1^b(W_1, W_2)$ and $H_2^b(W_1, W_2)$ associated with H_1 and H_2. We have that $H_1^b(W_1, W_2)$ induces $G(V_1, V_2)$, where $G(V_1, V_2)$ is some element of $\mathcal{G}$. Since $\mathcal{G}$ is degree-sequence-forcing, $H_2^b(W_1, W_2)$ also induces some element $G'(V_1', V_2')$ of $\mathcal{G}$. Let F be the element of $\mathcal{F}$ having $G'(V_1', V_2')$ as an associated bipartitioned graph. The only way that F may not be induced in H_2 is for H_2 to be C_5-free while F is not. However, if F induces C_5, then by assumption every element of $\mathcal{F}$ induces C_5, which implies that H_1 and hence H_2 induce C_5 as well. This is a contradiction, since H_2 then induces an element of $\mathcal{F}$. We conclude that $\{2K_2, C_4\} \cup \mathcal{F}$ is degree-sequence-forcing. $\square$

As we noted at the beginning of this subsection, the use of order-preserving parameters for general graphs as outlined in Remark 3.13 yields no new requirements of a potential non-minimal degree-sequence-forcing set. However, if we adapt the approach to the context of bipartitioned graphs, then we are able to obtain some necessary conditions on degree-sequence-forcing sets of bipartitioned graphs, as follows.

Proposition 3.50. *Every degree-sequence-forcing set $\mathcal{G}$ of bipartitioned graphs contains an element whose underlying graph is a forest.*

Proof. Let $\rho(H)$ denote the number of cycles in a graph H. Note that ρ is order-preserving: if F is an induced subgraph of H, then $\rho(F) \leq \rho(H)$. Let $\mathcal{G}$ be a set of bipartitioned graphs, and let $G(V_1, V_2)$ be an element of $\mathcal{G}$ whose underlying graph G minimizes ρ. If $\rho(G) > 0$, then let uv be an edge of G on a cycle, where $u \in V_1$ and $v \in V_2$. For vertices $x, y \notin V_1 \cup V_2$, define $V_1' = V_1 \cup \{x\}$ and $V_2' = V_2 \cup \{y\}$; define $H(V_1', V_2')$ to be the bipartitioned graph whose edge set consists of all edges of $G+xy$, plus the edge xy. Let $H'(V_1', V_2')$ be the bipartitioned graph resulting from the bipartitioned 2-switch $\{uv, xy\} \rightrightarrows \{uy, xv\}$ on $H(V_1', V_2')$. Note that $\rho(H') < \rho(G)$. Since ρ is order-preserving and $G(V_1, V_2)$ is minimal with respect to ρ, we have that $H'(V_1', V_2')$ is $\mathcal{G}$-free. Since $H(V_1', V_2')$ has the same degree sequence as $H'(V_1', V_2')$ and clearly induces an element of $\mathcal{G}$, we conclude that $\mathcal{G}$ is not degree-sequence-forcing. Thus every degree-sequence-forcing set of bipartitioned graphs contains an element $G(V_1, V_2)$ such that $\rho(G) = 0$, and the result follows. □

Proposition 3.51. *Every degree-sequence-forcing set $\mathcal{G}$ of bipartitioned graphs contains an element whose underlying graph is of the form $K_{\ell,m} + nK_2 + pK_1$ for $\ell, m, n, p \geq 0$.*

Proof. For any bipartitioned graph $H(V_1, V_2)$, let $\rho(H(V_1, V_2))$ denote the minimum number of edges that can be added to $H(V_1, V_2)$ so that the resulting underlying graph has the form $K_{\ell,m} + nK_2 + pK_1$ and is still bipartite with partite sets V_1, V_2. Note that if $F(W_1, W_2)$ is induced in $H(V_1, V_2)$, then $\rho(F(W_1, W_2)) \leq \rho(H(V_1, V_2))$. Now let $\mathcal{G}$ be a set of bipartitioned graphs, and let $G(V_1, V_2)$ be an element of $\mathcal{G}$ that minimizes ρ. Suppose that $\rho(G(V_1, V_2)) > 0$. Choose $u \in V_1$ and $v \in V_2$ such that uv belongs to a set of $\rho(G(V_1, V_2))$ edges, each having an endpoint in each of V_1 and V_2, that can be added to G to make it of the form $K_{\ell,m} + nK_2 + pK_1$. For vertices $x, y \notin V_1 \cup V_2$, define $V_1' = V_1 \cup \{x\}$ and $V_2' = V_2 \cup \{y\}$, and define $H(V_1', V_2')$ to be the bipartitioned graph whose edge set consists of all edges of G, plus the edges uy and xv. Let $H'(V_1', V_2')$ be the bipartitioned graph resulting from the bipartitioned 2-switch $\{xv, uy\} \rightrightarrows \{uv, xy\}$

on $H(V_1', V_2')$. It is easily seen that $\rho(H'(V_1', V_2')) < \rho(G(V_1, V_2))$. Since ρ is order-preserving and $G(V_1, V_2)$ is minimal with respect to ρ, we find that $H'(V_1', V_2')$ is $\mathcal{G}$-free. Since $H(V_1', V_2')$ has the same degree sequence as $H'(V_1', V_2')$ and clearly induces an element of $\mathcal{G}$, we conclude that $\mathcal{G}$ is not degree-sequence-forcing. Thus if $\mathcal{G}$ is a degree-sequence-forcing set of bipartitioned graphs, then some element $G(V_1, V_2)$ in $\mathcal{G}$ satisfies $\rho(G(V_1, V_2)) = 0$, and the result follows. $\square$

Corollary 3.52. *Every degree-sequence-forcing set of bipartitioned graphs contains two elements $G(V_1, V_2)$ and $H(W_1, W_2)$ such that $\overline{G(V_1, V_2)}$ has a forest for its underlying graph, and $\overline{H(W_1, W_2)}$ has an underlying graph of the form $K_{\ell,m} + nK_2 + pK_1$ for some $\ell, m, n, p \geq 0$.*

Proof. This follows from Remark 3.47 and Propositions 3.50 and 3.51. $\square$

Our first application of these results will be to characterize the degree-sequence-forcing singleton sets $\{G(V_1, V_2)\}$ of bipartitioned graphs.

Lemma 3.53. *Bipartitioned graphs $G(V_1, V_2)$ and $\overline{G(V_1, V_2)}$ both have the property that their underlying graphs are forests and graphs of the form $K_{\ell,m} + nK_2 + pK_1$ where $\ell, m, n, p \geq 0$ if and only if either $\min\{|V_1|, |V_2|\} \leq 1$ or $G \cong K_{1,m} + K_n$, where $1 \leq n \leq 2$.*

Proof. If $\min\{|V_1|, |V_2|\} \leq 1$ or $G \cong K_{1,m} + K_n$, where $1 \leq n \leq 2$, then $G(V_1, V_2)$ and its bicomplement satisfy the properties required. We now prove the converse. Let $G(V_1, V_2)$ and its bicomplement both have underlying graphs that have the forms specified. The graph G then has the form $K_{\ell,m} + nK_2 + pK_1$ for some $\ell, m, n, p \geq 0$ with $\ell \leq m$. Since G is also a forest, we have $0 \leq \ell \leq 1$.

Suppose first that $\ell = n = 0$. In this case, $G \cong (m + p)K_1$. Since the bicomplement of $G(V_1, V_2)$ is also a forest, either V_1 or V_2 contains at most one vertex.

If $\ell = 0$ and $n \geq 1$, then $G \cong nK_2 + (m+p)K_1$. Fix an edge uv in G. For any $x \in V_1 - \{u, v\}$ and $y \in V_2 - \{u, v\}$, we have x adjacent to y; otherwise, u, v, x, y belong to a component in $\overline{G(V_1, V_2)}$ that is not complete bipartite, a contradiction to our assumption. Thus either $\min\{|V_1|, |V_2|\} = 1$, or $m = p = 0$ and $n = 2$ and hence $G \cong 2K_2 \cong K_{1,1} + K_2$.

Suppose instead that $\ell = 1$. We may assume that $m \geq 2$ since otherwise we could write G as $n'K_2 + p'K_1$, which was handled in the previous case. Suppose that $\min\{|V_1|, |V_2|\} \geq 2$. We may also assume that the component $K_{1,m}$ has its center in V_2 (otherwise, the bicomplement of $G(V_1, V_2)$ contains a star component on 3 or more vertices whose center belongs to V_2, and we may proceed in the proof with the bicomplement). There is some vertex u in V_2 not belonging to the copy of $K_{1,m}$. If there is another vertex v in V_2 not belonging to the copy of $K_{1,m}$, then $\overline{G(V_1, V_2)}$ is not a forest; hence $|V_2| = 2$. Since G has the form $K_{1,m} + nK_2 + pK_1$, the vertex u has at most one neighbor in V_1. Any vertex in V_1 not contained in the copy of $K_{1,m}$ is adjacent to u; otherwise, u belongs to a component in $\overline{G(V_1, V_2)}$ that is not complete bipartite. Thus G is isomorphic to either $K_{1,m} + K_2$ or $K_{1,m} + K_1$. $\square$

Proposition 3.54. *The set $\mathcal{G} = \{G(V_1, V_2)\}$ is degree-sequence-forcing if and only if $G(V_1, V_2)$ satisfies one of the following:*

(i) $\min\{|V_1|, |V_2|\} \leq 1$*;*

(ii) $G \cong 2K_2$*;*

(iii) $G \cong K_{1,2} + K_n$*, where* $1 \leq n \leq 2$.

Proof. Let $\mathcal{G}$ be a degree-sequence-forcing set. By Propositions 3.50 and 3.51, Lemma 3.53, and Corollary 3.52, we have that either $\min\{|V_1|, |V_2|\} \leq 1$ or $G \cong K_{1,m} + K_n$ for $1 \leq n \leq 2$. Suppose first that $G \cong K_{1,m} + K_2$, with $\min\{|V_1|, |V_2|\} \geq$

2. We have $m > 0$; suppose that $m \geq 3$. Assume first that the center y of the copy of $K_{1,m}$ in G belongs to V_2. Let $V_2' = V_2 \cup \{x\}$, where $x \notin V_1 \cup V_2$, and form the bipartitioned graph $H_1(V_1, V_2')$ whose edge set consists of $E(G)$ plus an edge from x to a vertex in V_1 in each component of G. Let a be the neighbor of x belonging to the component of order 2 in G, and let b be a leaf of the copy of $K_{1,m}$ to which x is not adjacent in H_1. Form $H_2(V_1, V_2')$ by performing on $H_1(V_1, V_2')$ the bipartitioned 2-switch $\{xa, yb\} \rightrightarrows \{xb, ya\}$. Since $\mathcal{G}$ is degree-sequence-forcing, $H_2(V_1, V_2')$ induces $G(V_1, V_2)$, and to obtain a copy of $G(V_1, V_2)$ we must delete x, the only vertex of degree 2 in V_2'. However, deleting x yields an isolated vertex in $H_2(V_1, V_2')$, a contradiction, since G has no isolated vertex. A similar contradiction arises if we assume that y belongs to V_1; we simply interchange the roles of V_1 and V_2. Thus $m \leq 2$; hence $G \cong 2K_2$ or $G \cong K_{1,2} + K_2$.

Suppose next that $G \cong K_{1,m} + K_1$ with $\min\{|V_1|, |V_2|\} \geq 2$. We have $m > 0$, and the isolated vertex z in G belongs to the same partite set as the center y of the copy of $K_{1,m}$. Suppose that $m \geq 3$, and assume that y and z belong to V_2. Let $V_2' = V_2 \cup \{x\}$ and $V_1' = V_1 \cup \{a\}$, where $x, a \notin V_1 \cup V_2$, and let b, c be two neighbors of y in V_1. Form the bipartitioned graph $H_1(V_1', V_2')$ whose edge set consists of $E(G)$ plus the edges xc, xa, and za. Form $H_2(V_1', V_2')$ by performing on $H_1(V_1', V_2')$ the bipartitioned 2-switch $\{yb, xa\} \rightrightarrows \{ya, xb\}$. For $H_2(V_1', V_2')$ to induce $G(V_1, V_2)$, the vertex y must be the center of the copy of $K_{1,m}$, as it is the only vertex with degree greater than 2; the neighbors of y must be the leaves of the copy of $K_{1,m}$. However, both x and z are adjacent to a neighbor of y, so $G(V_1, V_2)$ is not induced in $H_2(V_1', V_2')$, a contradiction. A similar argument produces a contradiction when y and z belong to V_1. We conclude again that $m \leq 2$, which produces the desired result.

We have shown that the degree-sequence-forcing set $\mathcal{G}$ must satisfy the conditions stated in the proposition. To see that these conditions are sufficient for $\mathcal{G}$

to be degree-sequence-forcing, we apply Theorem 3.49 to the C_5-inducing graphs F from Corollary 3.35, Proposition 3.37, and Corollary 3.44. Every bipartitioned graph of the form described in the proposition appears as the associated bipartitioned graph of some such F. $\square$

In preparation for later results we now present a proposition on the structure of split graphs.

Proposition 3.55. *Suppose that S is a split graph with more than one associated bipartitioned graph, up to isomorphism. It follows that S has exactly two associated bipartitioned graphs $G(V_1, V_2)$ and $H(W_1, W_2)$, up to isomorphism, with $|V_2|$ equal to the clique number $\omega(S)$ and $|W_2| = \omega(S) - 1$. The graph $G(V_1, V_2)$ has some isolated vertex $u \in V_2$, and the graph $H(W_1, W_2)$ has some vertex $v \in W_1$ that dominates W_2 such that $G(V_1, V_2) - u \cong H(W_1, W_2) - v$.*

Proof. Let S be a split graph. Suppose that $G(V_1, V_2)$ and $H(W_1, W_2)$ are two bipartitioned graphs associated with S, where $|V_2| = |W_2|$. We show that $G(V_1, V_2) \cong H(W_1, W_2)$. We may assume without loss of generality that $V_1 \cup V_2 = W_1 \cup W_2 = V(S)$. Since the independent set W_1 can intersect the clique V_2 in at most one vertex, we have $|V_2 \cap W_2| \geq |V_2| - 1$. If $|V_2 \cap W_2| = |V_2|$, then $V_2 = W_2$ and in fact $G(V_1, V_2) = H(W_1, W_2)$. Suppose instead that $|V_2 \cap W_2| = |V_2| - 1$. We may write $V_2 - W_2 = \{v\}$ and $W_2 - V_2 = \{w\}$, and we have $v \in W_1$ and $w \in V_1$. Since V_1 and W_1 are independent sets, we find that $N_S(v) \subseteq W_2$ and $N_S(w) \subseteq V_2$. Since $N_S(v)$ and $N_S(w)$ both contain $V_2 \cap W_2$, the map $\phi : V(S) \to V(S)$ that transposes v and w and fixes all other vertices in S is an automorphism such that $\phi(V_2) = W_2$. This same map translates to an isomorphism between bipartitioned graphs. Thus we have shown that if $|V_2| = |W_2|$, then $G(V_1, V_2) \cong H(W_1, W_2)$.

Suppose now that $G(V_1, V_2)$ and $H(W_1, W_2)$ are two nonisomorphic bipartitioned graphs associated with S. We have $|V_2| \neq |W_2|$; assume without loss of generality that $|V_2| > |W_2|$. Since at most one vertex of a maximum clique can belong to W_1, we have $|W_2| \geq \omega(S) - 1$; hence $|V_2| = \omega(S)$ and $|W_2| = \omega(S) - 1$. Let Q be a clique of size $\omega(S)$ in S. Since W_1 is an independent set in S, at most one vertex of Q can belong to W_1. It follows that we may write $Q = W_2 \cup \{q\}$, where q is some vertex in W_1. Note that q is adjacent to every vertex in W_2 in S and hence in H. Since W_1 is an independent set, q has no other neighbors in S. Thus $(W_1 - \{q\}, W_2 \cup \{q\})$ is a splitting partition of $V(S)$. Let $H^*(W_1 - \{q\}, W_2 \cup \{q\})$ be the bipartitioned graph associated with this partition. The vertex q is an isolated vertex in $W_2 \cup \{q\}$, and $|W_2 \cup \{q\}| = \omega(S)$. As we showed above, $H^*(W_1 - \{q\}, W_2 \cup \{q\})$ is isomorphic to $G(V_1, V_2)$. Let q' be the image of q under an isomorphism from $H^*(W_1 - \{q\}, W_2 \cup \{q\})$ to $G(V_1, V_2)$. The vertex q' is an isolated vertex in V_2 whose deletion from $G(V_1, V_2)$ yields a bipartitioned graph isomorphic to $H(W_1, W_2) - q$, as desired. $\square$

As a consequence of Proposition 3.55, if a split graph has two associated bipartitioned graphs, we may express them in the form $G(V_1, V_2 + u)$ and $H(V_1 + u, V_2)$ for some graphs G, H on the same vertex set.

Proposition 3.56. *Let $\mathcal{F} = \{G(V_1, V_2 + u), H(V_1 + u, V_2)\}$ be the set of associated bipartitioned graphs of a split graph S. Let $G' = G - u$ and $H' = H - u$. If $\mathcal{F}$ is a degree-sequence-forcing pair of bipartitioned graphs, then $G'(V_1, V_2) \cong H'(V_1, V_2)$ is one of the following:*

(i) *nK_1, with $n \geq 0$ and $|V_2| \leq 1$,*

(ii) *K_2,*

(iii) *$K_{1,2}$,*

(iv) *$K_2 + K_1$, with $|V_2| = 1$,*

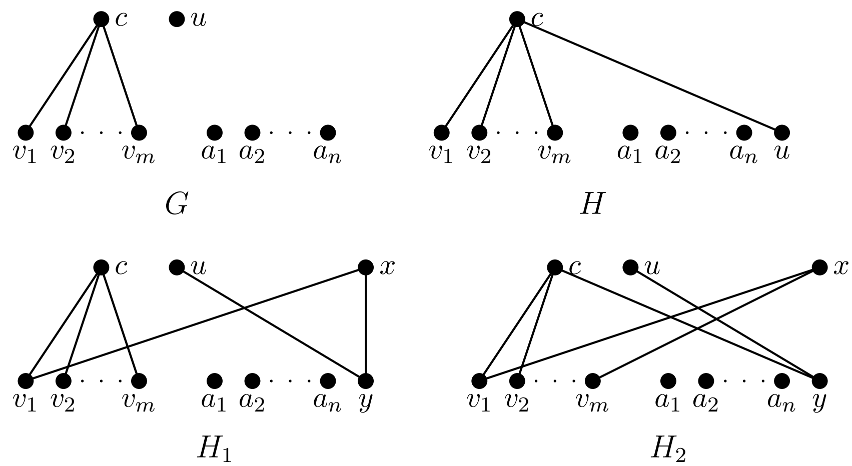


Figure 3.10: Bipartitioned graphs from Subcase 1b of Proposition 3.56.

(v) $K_{1,2} + K_1$, *with* $|V_2| = 1$,

(vi) *the bicomplement of one of the graphs above.*

Proof. By Proposition 3.55 we may assume that $\mathcal{F}$ has the form $\mathcal{F} = \{G(V_1, V_2 + u), H(V_1 + u, V_2)\}$ and that $G'(V_1, V_2) \cong H'(V_1, V_2)$ as bipartitioned graphs, where $G' = G - u$ and $H' = H - u$. By Propositions 3.50 and 3.51, Corollary 3.52, and the fact that the classes of forests, bicomplements of forests, graphs of the form $K_{\ell,m} + nK_2 + pK_1$, and bicomplements of these last graphs are hereditary under induced subgraphs, we find that $G'(V_1, V_2)$ (and hence $H'(V_1, V_2)$) must be one of the graphs mentioned in Lemma 3.53.

Case 1: $\min\{|V_1|, |V_2|\} \leq 1$.

Subcase 1a: $\min\{|V_1|, |V_2|\} = 0$. In this case all vertices belong to one part of the bipartition in both G' and H', and $\mathcal{F}$ is clearly degree-sequence-forcing.

Subcase 1b: $\min\{|V_1|, |V_2|\} = 1$. We assume that $|V_2| = 1$, since if $|V_1| = 1$ then the bicomplement of $G'(V_1, V_2)$ falls under this case. With $|V_2| = 1$, we find that $G'(V_1, V_2) \cong H'(V_1, V_2) \cong K_{1,m} + nK_1$, where $m, n \geq 0$.

Claim 1: If $m \geq 3$ *and* $n \geq 0$, *then* $\{G(V_1, V_2 + u), H(V_1 + u, V_2)\}$ *is not degree-sequence-forcing.*

Proof. Graphs G and H are as shown in Figure 3.10. Let c denote the center of the nontrivial star component in G, let $v_1, \ldots, v_m$ denote the leaves adjacent to c, and let $a_1, \ldots, a_n$ denote the isolated vertices in V_1. Form $H_1(V_1 + y, V_2 + u + x)$ by adding to G vertices x and y and edges xv_1, xy, uy, as shown in Figure 3.10. Let $H_2(V_1 + y, V_2 + u + x)$ be the bipartitioned graph resulting from the 2-switch $\{cv_m, xy\} \rightrightarrows \{xv_m, cy\}$. Suppose that a copy of $G(V_1, V_2 + u)$ is induced in $H_2(V_1 + y, V_2 + u + x)$. We may obtain this copy by deleting a vertex in each of $V_1 + y$ and $V_2 + u + x$. Because of degree and distance conditions, we cannot delete c and hence must delete both u and x, a contradiction. We also see that H is not induced in H_2. Thus $\{G(V_1, V_2 + u), H(V_1 + u, V_2)\}$ is not degree-sequence-forcing.

Claim 2: If $m \in \{1, 2\}$ and $n \geq 2$, then $\{G(V_1, V_2 + u), H(V_1 + u, V_2)\}$ is not degree-sequence-forcing.

Proof. Graphs G and H are as shown in Figure 3.11 (we have illustrated the case $m = 2$). Again let c denote the center of the nontrivial star component in G, let $v_1, \ldots, v_m$ denote the leaves adjacent to c, and let $a_1, \ldots, a_n$ denote the isolated vertices in V_1. As shown in Figure 3.11, form $H_1(V_1 + y, V_2 + u + x)$ by adding to G vertices x and y and edges $xy, xa_1, xa_2, \ldots, xa_n, uy$; also add edge xv_2 if $m = 2$. Form the bipartitioned graph $H_2(V_1 + y, V_2 + u + x)$ by performing on H_1 the 2-switch $\{cv_1, xy\} \rightrightarrows \{cy, xv_1\}$. Suppose that $H_2(V_1 + y, V_2 + u + x)$ induces a copy of $G(V_1, V_2 + u)$. We may obtain this copy (call it $G''(W_1, W_2)$) by deleting one vertex in H_2 from each of $V_1 + y$ and $V_2 + u + x$. In order to leave n isolated vertices in $W_1 \cap (V_1 + y)$, we must delete x; in order to leave an isolated vertex in $W_2 \cap (V_2 + u + x)$, we must delete y. However, no vertex in W_2 would then have degree at least m in G'', a contradiction. Furthermore, if a copy of $H(V_1 + u, V_2)$ were induced in $H_2(V_1 + y, V_2 + u + x)$, then deleting two vertices of $V_2 + u + x$ would yield this subgraph, and for no pair is this the case. Thus

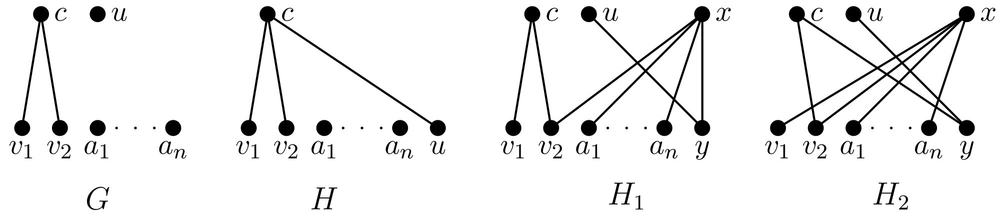


Figure 3.11: Bipartitioned graphs from Subcase 1b of Proposition 3.56.

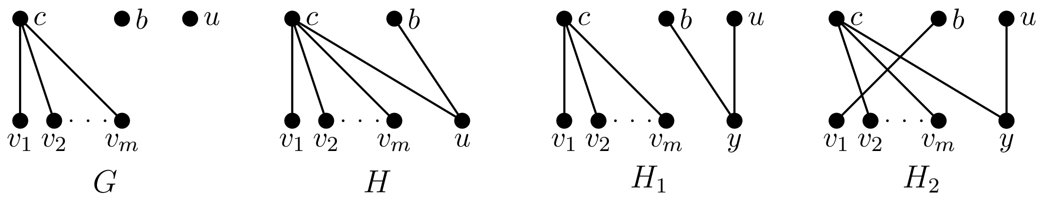


Figure 3.12: Bipartitioned graphs from Subcase 2a of Proposition 3.56.

$\{G(V_1, V_2 + u), H(V_1 + u, V_2)\}$ is not degree-sequence-forcing.

Case 2: $G'(V_1, V_2) \cong H'(V_1, V_2) \cong K_{1,m} + K_n$ for $1 \leq n \leq 2$. If $m + n \leq 2$ then Case 1 applies, so we assume that $m + n \geq 3$.

Subcase 2a: $G'(V_1, V_2) \cong H'(V_1, V_2) \cong K_{1,m} + K_1$, where $m \geq 2$. Graphs G and H are as shown in Figure 3.12. Form $H_1(V_1+y, V_2+u)$ by adding to G vertex y and edges yb, yu. Let $H_2(V_1 + y, V_2 + u)$ be the bipartitioned graph resulting from the 2-switch $\{cv_m, by\} \rightrightarrows \{cy, bv_m\}$. If a copy of $G(V_1, V_2 + u)$ is induced in $H_2(V_1 + y, V_2 + u)$, it may be obtained by deleting a vertex in $V_1 + y$. Since c is the only vertex in $V_2 + u$ having degree m, none of its neighbors may be the deleted vertex; however, deleting v_m leaves a subgraph not isomorphic to G. No vertex of H_2 has degree $m + 1$, so $H(V_1 + u, V_2)$ is also not an induced subgraph of $H_2(V_1+y, V_2+u)$. Thus $\{G(V_1, V_2+u), H(V_1+u, V_2)\}$ is not degree-sequence-forcing.

Subcase 2b: $G'(V_1, V_2) \cong H'(V_1, V_2) \cong K_{1,m} + K_2$, where $m \geq 1$. Graphs G and H must be as shown in Figure 3.13. Form $H_1(V_1 + u + y, V_2 + x)$ by adding to H the vertices x, y and edges xv for $v \in \{v_1, v_2, \ldots, v_m, y\}$. Obtain

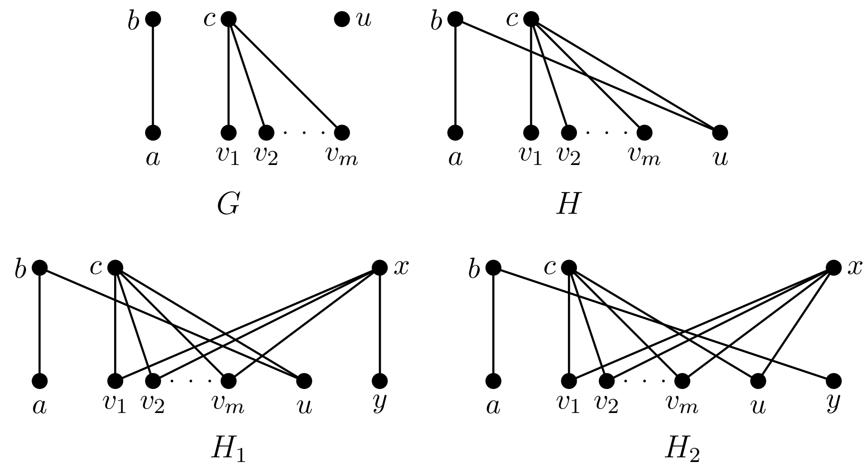


Figure 3.13: Bipartitioned graphs from Subcase 2b of Proposition 3.56.

$H_2(V_1 + u + y, V_2 + x)$ by performing on H_1 the 2-switch $\{xy, bu\} \rightrightarrows \{xu, by\}$. If a copy of $H(V_1 + u, V_2)$ is induced in $H_2(V_1 + u + y, V_2 + x)$, we may isolate it by deleting from H_2 one vertex in each of $V_1 + u + y, V_2 + x$. However, H is connected, and there is no suitable pair of vertices in H_2 that may be deleted to leave a connected subgraph. Thus $H_2(V_1 + u + y, V_2 + x)$ is $\{H(V_1 + u, V_2)\}$-free. Graph $H_2(V_1+u+y, V_2+x)$ is also $\{G(V_1, V_2+u)\}$-free, since no two vertices may be deleted to leave in $V_1 + u + y$ exactly two vertices of degree 1 with different neighbors. Thus $\{G(V_1, V_2+u), H(V_1+u, V_2)\}$ is not degree-sequence-forcing. □

We conclude with the characterization of all non-minimal degree-sequence-forcing triples of graphs.

Proof of Theorem 3.27. By the results of Section 3.4.1, it suffices to show that every degree-sequence-forcing triple $\mathcal{F}$ is listed in the statement of the theorem. As indicated by items 1 and 2(i) in the theorem, we may assume that $\mathcal{F}$ has the form $\{2K_2, C_4, F\}$, where F is $\{2K_2, C_4\}$-free. It follows from Theorem 3.49 that to characterize F such that $\mathcal{F}$ is degree-sequence-forcing, it suffices to characterize the degree-sequence-forcing sets $\mathcal{G}$ of bipartitioned graphs such that $\mathcal{G}$ consists of the bipartitioned graph or graphs associated with a single pseudo-split graph F.

Propositions 3.54 and 3.56 provide requirements on the structure of F, which we now examine in detail. Suppose first that F has a unique pseudo-splitting partition. This implies that F induces C_5, or F^s has a unique partition into a clique and independent set. In either case $\mathcal{G}$ consists of a single graph, and from Proposition 3.54 it follows that F is one of the graphs listed in items 2(iii), 2(iv), 2(vi), and 2(viii) in the statement of the theorem. Suppose instead that F has more than one pseudo-splitting partition. This happens only if F is split, and it implies that $\mathcal{G}$ consists of two graphs as described in Proposition 3.56. It follows that F is one of the graphs listed in items 2(ii), 2(v), 2(vii), or 2(ix) in the theorem. □

3.5 Minimal degree-sequence-forcing sets

In this section we present results on minimal degree-sequence-forcing sets, those degree-sequence-forcing sets that contain no proper subset that is also degree-sequence-forcing. As we observed in Section 3.4.2, arguments that use order-preserving parameters to impose conditions on a degree-sequence-forcing set are neutralized when the set considered contains a degree-sequence-forcing proper subset; this approach yields no information on the other elements of the larger degree-sequence-forcing set. In contrast, as we now show, every element of a minimal degree-sequence-forcing set is subject to bounds on the numbers of vertices and edges that it may contain.

Proposition 3.57. *Let $\mathcal{F} = \{F_1, \ldots, F_k\}$, where the graphs in $\mathcal{F}$ are indexed in order of the sizes of their vertex sets, from smallest to largest. If $\mathcal{F}$ is a minimal degree-sequence-forcing set, then $|V(F_{i+1})| - |V(F_i)| \leq 2$ for all $i \in \{1, \ldots, k-1\}$.*

Proof. Let $\mathcal{F}$ be a minimal degree-sequence-forcing set with elements indexed as described. For any $i \in \{1, \ldots, k-1\}$, let $\mathcal{G}$ be the set $\{F_1, \ldots, F_i\}$. Since $\mathcal{F}$

is minimal, $\mathcal{G}$ is not degree-sequence-forcing. By Proposition 3.11 there exists a $\mathcal{G}$-breaking pair (H, H') such that $|V(H')| \leq |V(F_i)| + 2$. Since $\mathcal{F}$ is degree-sequence-forcing, H' induces an element F_j of $\mathcal{F}$; note that $j > i$. Thus

$$|V(F_{i+1})| - |V(F_i)| \leq |V(F_j)| - |V(F_i)| \leq |V(H')| - |V(F_i)| \leq 2,$$

as claimed. $\square$

Proposition 3.58. *Let $\mathcal{F} = \{F'_1, \ldots, F'_k\}$, where the graphs in $\mathcal{F}$ are indexed in order of the sizes of their edge sets, from smallest to largest. If $\mathcal{F}$ is a minimal degree-sequence-forcing set, then*

$$|E(F'_{i+1})| \leq \max_{j \leq i} \left\{ |E(F'_j)| + 2|V(F'_j)| \right\}$$

for all $i \in \{1, \ldots, k-1\}$.

Proof. Let $\mathcal{F}$ be a minimal degree-sequence-forcing set with elements indexed as described. For any $i \in \{1, \ldots, k-1\}$, let $\mathcal{G}'$ be the set $\{F'_1, \ldots, F'_i\}$. Since $\mathcal{F}$ is minimal, $\mathcal{G}'$ is not degree-sequence-forcing. As we see in the proof of Proposition 3.11 there exists a $\mathcal{G}'$-breaking pair (H, H') such that every vertex of H not belonging to a chosen induced copy of an element F'_j of $\mathcal{G}'$ belongs to one of the edges involved in the 2-switch that changes H into H'; there are at most two such vertices. These vertices are incident with all edges of H that do not belong to the chosen copy of F'_j; thus H contains at most $|E(F'_j)| + 2|V(F'_j)|$ edges (note that any vertex not in the copy of F'_j is involved in the 2-switch changing H into H' and hence is not a dominating vertex in H). Note that H' has the same number of edges as H, and since $\mathcal{F}$ is degree-sequence-forcing, H' induces an element F'_ℓ

of $\mathcal{F}$ where $\ell > i$. It follows that

$$|E(F'_{i+1})| \leq |E(F'_\ell)| \leq |E(H')| \leq |E(F'_j)| + 2|V(F'_j)|,$$

which completes the proof. $\square$

These restrictions imply that degree-sequence-forcing sets with few elements contain graphs that do not differ greatly in the numbers of vertices and edges they contain. It also implies that small degree-sequence-forcing sets contain small graphs, as the next result shows.

Theorem 3.59. *If $\mathcal{F}$ is a minimal degree-sequence-forcing set of k graphs, then the number of vertices in any element of $\mathcal{F}$ is at most*

$$4k - \frac{3}{2} + \sqrt{12k^2 - 10k + \frac{1}{4}};$$

hence there are finitely many minimal degree-sequence-forcing k-sets.

Proof. Let $\mathcal{F}$ be a minimal degree-sequence-forcing set of k graphs, and denote the graphs of $\mathcal{F}$ both by $\{F_1, \ldots, F_k\}$ and by $\{F'_1, \ldots, F'_k\}$, where the graphs are indexed in order of the sizes of their vertex sets and edge sets, respectively, from smallest to largest. Let $n_1 = |V(F_1)|$. As a consequence of Proposition 3.57, we observe that

$$|V(F_k)| \leq n_1 + 2(k-1).$$

Using this result and Proposition 3.58, we find that

$$\begin{aligned}
|E(F'_j)| &\leq \max_{i<j} \{|E(F'_i)| + 2|V(F'_i)|\} \\
&\leq |E(F'_{j-1})| + 2|V(F_k)| \\
&\leq |E(F'_{j-1})| + 2n_1 + 4(k-1).
\end{aligned}$$

Summing the values of $|E(F'_j)| - |E(F'_{j-1})|$ as j ranges from 2 to k yields

$$|E(F'_k)| - |E(F'_1)| \leq 2(k-1)n_1 + 4(k-1)^2. \tag{3.1}$$

We thus arrive at an upper bound on the difference in the number of edges between any two graphs in the set $\mathcal{F}$. To find a lower bound on this difference, we recall from Proposition 3.12 and Corollary 3.14 that some element F_s of $\mathcal{F}$ is a forest of stars, while some element F_c of $\mathcal{F}$ is the complement of a forest of stars. Recall that an n-vertex graph has $n(n-1)/2$ edges, and a forest on n vertices has at most $n-1$ edges, so the number of edges in the complement of a forest is at least $n(n-1)/2 - (n-1)$, which simplifies to $(n-2)(n-1)/2$. We have

$$\begin{aligned}
|E(F'_k)| - |E(F'_1)| &\geq |E(F_c)| - |E(F_s)| \\
&\geq \frac{(n_1-2)(n_1-1)}{2} - (|V(F_k)| - 1) \\
&\geq \frac{(n_1-2)(n_1-1)}{2} - (n_1 + 2(k-1) - 1)\,.
\end{aligned}$$

Combining this inequality and the one in (3.1), we find that

$$\frac{(n_1-2)(n_1-1)}{2} - (n_1 + 2(k-1) - 1) \leq 2(k-1)n_1 + 4(k-1)^2,$$

which reduces to

$$n_1 \leq 2k + \frac{1}{2} + \sqrt{12k^2 - 10k + \frac{1}{4}}.$$

Since $|V(F_k)| \leq n_1 + 2(k-1)$, the result follows. $\square$

The bound on $|V(F_k)|$ in Theorem 3.59 does not appear to be tight. The theorem implies that the largest graphs that minimal degree-sequence-forcing singletons and pairs can contain have at most four and at most nine vertices, respectively, when in fact the largest graph in any degree-sequence-forcing singleton has two vertices, and the largest graph in any minimal degree-sequence-forcing pair has five vertices.

As an illustration of Theorem 3.59, observe that Theorems 3.17 and 3.18 show that there are finitely many minimal degree-sequence-forcing singletons and pairs. Theorems 3.18 and 3.27 demonstrate that the condition of minimality is necessary, as there are infinitely many non-minimal degree-sequence-forcing pairs and triples.

Although for any natural number k there are finitely many minimal degree-sequence-forcing k-sets, we observe that there are infinitely many minimal degree-sequence-forcing sets. To see this, note that for any natural number n the set $\mathcal{F}$ of all graphs on n vertices is a degree-sequence-forcing set: given a graphic sequence π, a realization of π induces an element of $\mathcal{F}$ if and only if π has at least n entries. The set $\mathcal{F}$ may not be a minimal degree-sequence-forcing set, but by definition it must contain such a subset. Thus for any natural number n there exists a minimal degree-sequence-forcing set $\mathcal{F}$ containing only n-vertex graphs. Theorem 3.59 implies a lower bound on the size of such a set. Thus minimal degree-sequence-forcing sets can be arbitrarily large. We conclude this section with a question: Does there exist an infinite minimal degree-sequence-forcing set?

3.6 Edit-leveling sets

Our motivation for defining degree-sequence-forcing sets came, in part, from the characterizations that split graphs have in terms of their degree sequences and forbidden subgraphs. The split graphs also provide the motivation for the final topic of this chapter on degree-sequence-forcing sets. In the paper [20] in which they gave a degree sequence characterization of split graphs, Hammer and Simeone defined the *splittance* $\sigma(G)$ of a graph G to be the minimum number of edges that can be added to or deleted from G to produce a split graph. The split graphs are precisely those graphs G for which $\sigma(G) = 0$. Hammer and Simeone showed that the splittance of a graph G may be computed directly from the degree sequence of G.

Theorem 3.60 ([20]). *Let G be a graph with degree sequence $(d_1, d_2, \ldots, d_n)$ such that $d_1 \geq d_2 \geq \cdots \geq d_n$. Let $m = \max\{k : 1 \leq k \leq n \text{ and } d_k \geq k - 1\}$. The splittance of G is given by*

$$\sigma(G) = \frac{1}{2}\left(m(m-1) - \sum_{i=1}^{m} d_i + \sum_{i=m+1}^{n} d_i\right).$$

Thus, if $\mathcal{F} = \{2K_2, C_4, C_5\}$, then $\mathcal{F}$ has the property that for every graph G the degree sequence not only determines whether G is $\mathcal{F}$-free, but also provides an exact measure of how far G is from being $\mathcal{F}$-free. We seek to generalize this property.

For a graph G and a graph class $\mathcal{P}$, the *edit distance from G to $\mathcal{P}$*, denoted $\mathrm{dist}(G, \mathcal{P})$, is the minimum number of edges that can be added or deleted to G to produce an element of $\mathcal{P}$ (if this is possible); in other words,

$$\mathrm{dist}(G, \mathcal{P}) = \min\{|E(G) \triangle E(G')| : G' \in \mathcal{P} \text{ and } |V(G)| = |V(G')|\},$$

where $A \triangle B$ denotes the symmetric difference of sets A and B. If $\mathcal{P}$ contains no graphs on $|V(G)|$ vertices, then we define $\mathrm{dist}(G, \mathcal{P}) = \infty$.

Define a graph class $\mathcal{P}$ to be *edit-level* if for every graph sequence π and two realizations G and G' of π we have $\mathrm{dist}(G, \mathcal{P}) = \mathrm{dist}(G', \mathcal{P})$. Thus the degree sequence of a graph uniquely determines the edit distance from the graph to an edit-level graph class. Define a set $\mathcal{F}$ of graphs to be *edit-leveling* if the class of $\mathcal{F}$-free graphs is edit-level.

With these definitions, the set $\{2K_2, C_4, C_5\}$ is edit-leveling. Furthermore, for each natural number n the set $\mathcal{F}$ of all graphs on n vertices is edit-leveling, since the edit distance from a graph G to the $\mathcal{F}$-free graphs is either 0 or ∞, depending on how many vertices G contains, and this can be determined immediately from the degree sequence of G. Thus there are infinitely many edit-leveling sets. We present another example of one.

Proposition 3.61. *For any natural number k, let*

$$\mathcal{F}_k = \{F : k \leq |E(F)| \leq k + \delta(F) - 1\}.$$

The set $\mathcal{F}_k$ is edit-leveling.

Proof. If G has at most $k-1$ edges, then G is $\mathcal{F}_k$-free. We show that the inverse of this statement is true. If G has k or more edges, then we iteratively delete vertices from G that leave the remaining graph with at least k edges until this is no longer possible. Let G' be the resulting induced subgraph. Since deleting any vertex from G' yields a graph with fewer than k edges, $|E(G')| - \delta(G') < k$; thus G' has at most $k + \delta(G') - 1$ edges. We conclude that G induces a subgraph G' isomorphic to an element of $\mathcal{F}_k$.

By the Degree Sum Formula, we may determine from the degree sequence of a graph how many edges it has; thus if G is a graph with degree sequence $(d_1, \ldots, d_n)$, then the edit distance from G to the class of $\mathcal{F}_k$-free graphs is given by $\frac{1}{2}\sum d_i - (k-1)$. Since the edit distance depends only on the degree sequence of G and not on G itself, $\mathcal{F}_k$ is edit-leveling. $\square$

If $\mathcal{F}$ is any set of graphs and $\mathcal{P}$ is the class of $\mathcal{F}$-free graphs, then for every nonnegative integer k define $\mathcal{P}^{(k)}$ to be the set of graphs at edit-distance at most k from $\mathcal{P}$. Observe that $\mathcal{P}^{(k)}$ is a hereditary family of graphs, so it can be characterized in terms of a set of minimal forbidden subgraphs; let $\mathcal{F}^{(k)}$ denote this set.

As an example, if $\mathcal{F} = \{K_2\}$ and $\mathcal{P}$ is the set of $\mathcal{F}$-free (that is, edgeless) graphs, then $\mathcal{P}^{(k)}$ is set of graphs on at most k edges, so $\mathcal{F}^{(k)} = \mathcal{F}_{k+1}$, another edit-leveling set. In general, we have the following.

Proposition 3.62. *If $\mathcal{F}$ is an edit-leveling set of graphs, then for every nonnegative integer k the set $\mathcal{F}^{(k)}$ is edit-leveling.*

Proof. Let $\mathcal{P}$ denote the set of $\mathcal{F}$-free graphs. The statement follows from observing that $\text{dist}(G, \mathcal{P}^{(k)}) = \max\{\text{dist}(G, \mathcal{P}) - k, 0\}$ for every graph G, and the quantity on the right-hand side of the equation is uniquely determined by the degree sequence of G, since $\mathcal{F}$ is assumed to be edit-leveling. □

For an edit-level family $\mathcal{P}$ of graphs, knowing the degree sequence of a graph G is enough to determine whether $\text{dist}(G, \mathcal{P}) = 0$, that is, whether $G \in \mathcal{P}$. Hence edit-level families are degree-determined, and edit-leveling sets are necessarily degree-sequence-forcing. Not every degree-sequence-forcing set is edit-leveling, however. For example, let $\mathcal{F}$ be the set $\{2K_2, C_4\}$, and let $\mathcal{P}$ be the set of $\{2K_2, C_4\}$-free graphs. The graphs $C_5 + K_2$ and $C_4 + P_3$ are both realizations of the degree sequence $(2, 2, 2, 2, 2, 2, 1, 1)$. We have $\text{dist}(C_5 + K_2, \mathcal{P}) = 1$, since $C_5 + K_2$ induces $2K_2$ but $C_5 + 2K_1$ is $\{2K_2, C_4\}$-free and may be obtained by deleting an edge from $C_5 + K_2$. We have $\text{dist}(C_4 + P_3, \mathcal{P}) > 1$, since no single edge may be added or deleted from $C_4 + P_3$ to produce a $\{2K_2, C_4\}$-free graph. Hence $\{2K_2, C_4\}$ is not edit-leveling.

As our final result of this section, we give a characterization of edit-leveling sets in terms of degree-sequence-forcing sets.

Proposition 3.63. *A set $\mathcal{F}$ is edit-leveling if and only if the set $\mathcal{F}^{(k)}$ is degree-sequence-forcing for every nonnegative integer k.*

Proof. Let $\mathcal{P}$ be the set of $\mathcal{F}$-free graphs. Suppose first that $\mathcal{F}$ is edit-level. Let G and G' be any two graphs having the same degree sequence. Fix a nonnegative integer k, and suppose that G is $\mathcal{F}^{(k)}$-free. We have

$$\begin{aligned}
\operatorname{dist}(G', P^{(k)}) &= \max\{\operatorname{dist}(G', \mathcal{P}) - k, 0\} \\
&= \max\{\operatorname{dist}(G, \mathcal{P}) - k, 0\} \\
&= \operatorname{dist}(G, P^{(k)}) \\
&= 0,
\end{aligned}$$

so G' is also $\mathcal{F}^{(k)}$-free. It follows that $\mathcal{F}^{(k)}$ is degree-sequence-forcing.

Suppose now that $\mathcal{F}$ is not edit-level. This implies the existence of two graphs G and G' having the same degree sequence for which $\operatorname{dist}(G, \mathcal{P}) < \operatorname{dist}(G', \mathcal{P})$. Let $k = \operatorname{dist}(G, \mathcal{P})$. Note that G belongs to $\mathcal{P}^{(k)}$ and is hence $\mathcal{F}^{(k)}$-free; note also that $\operatorname{dist}(G', \mathcal{P}^{(k)}) = \operatorname{dist}(G', \mathcal{P}) - k > 0$, so G' is not $\mathcal{F}^{(k)}$-free. Thus (G', G) is an $\mathcal{F}^{(k)}$-breaking pair, and $\mathcal{F}^{(k)}$ is not degree-sequence-forcing. $\square$

CHAPTER 4

The A_4-structure of a graph

4.1 Introduction

Given a simple graph G, the P_4-*structure* of G is the 4-uniform hypergraph with the same vertex set as G whose edges are the vertex subsets inducing 4-vertex paths. Chvátal [13] defined the P_4-structure in 1984 in studying the complexity of recognizing perfect graphs. Since its introduction, the P_4-structure has also been used in refinements of the modular decomposition of a graph (see [29] and [46]) and in defining or characterizing several classes of graphs (see [11] for a hierarchy of several graph classes defined in terms of their P_4-structure).

If $\mathcal{F}$ is any set of unlabeled graphs, we may similarly define the $\mathcal{F}$-structure of a graph G as the hypergraph on the vertex set of G having as edges the vertex subsets on which G induces elements of $\mathcal{F}$. Such structures have been considered for the cases where $\mathcal{F}$ is $\{P_3\}$, $\{C_5, \text{paw}, P_3 + K_1\}$, $\{2K_2, C_4, C_5\}$, $\{P_3, K_2 + K_1\}$, and $\{K_3, 3K_1\}$ (see [24–28]). A *realization* of an $\mathcal{F}$-structure H is a graph whose $\mathcal{F}$-structure is H, up to hypergraph isomorphism.

In this chapter we consider the A_4-*structure* of a graph G, which we define as the 4-uniform hypergraph on the vertex set of G having as edges those vertex subsets that induce an element of $\{2K_2, C_4, P_4\}$. The name for this hypergraph comes from the fact that $2K_2, C_4$, and P_4 are the 4-vertex graphs that have an alternating 4-cycle, as shown in Chapter 3. Consider an alternating 4-cycle on vertex set $\{a, b, c, d\}$ in the graph G, such that $ab, cd \in E(G)$ and $bc, ad \notin E(G)$.

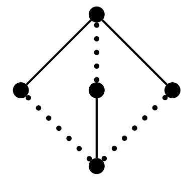


Figure 4.1: The configuration $\mathcal{C}$.

We will denote such a configuration by $[a, b : c, d]$.

Our motivation for studying this hypergraph comes from several sources. First, alternating 4-cycles are a fundamental notion in the study of degree sequences. Recall our definition of a 2-switch from Section 3.2 and in particular the result of Fulkerson, Hoffman, and McAndrew [17] cited in Theorem 3.8. Since alternating 4-cycles play an important role in the study of realizations of a degree sequence, we might expect to find relationships between the A_4-structure and the degree sequence of a graph.

As a second motivation, we note that alternating 4-cycles have been used in defining or characterizing several interesting classes of graphs. For example, the *threshold graphs* are precisely those graphs containing no alternating 4-cycle [14], *matroidal graphs* are those graphs for which the pairs of edges inducing alternating 4-cycles are exactly the circuits of a matroid on the edge set of the graph [42], and *matrogenic graphs* are the graphs for which the vertex sets of induced copies of $2K_2$, C_4, and P_4 form the circuits of a matroid on the vertex set of the graph [15].

Matroidal and matrogenic graphs also have characterizations in terms of other forbidden structures. Matroidal graphs were characterized in [42] as those graphs that do not contain an induced 5-cycle or the configuration $\mathcal{C}$ shown in Figure 4.1. Matrogenic graphs were characterized in [15] as those graphs that forbid $\mathcal{C}$ (but allow induced 5-cycles).

Characterizations exist in terms of the A_4-structure for threshold, matroidal, and matrogenic graphs. Examining the A_4-structures of all graphs on five vertices, we see that C_5 is the only one having more than three edges in its A_4-structure, and that those in which $\mathcal{C}$ occurs are the ones whose A_4-structures have exactly two or three edges.

Observation 4.1. *A graph is a threshold graph if and only if its A_4-structure contains no edges. A graph is matroidal if and only if no five of its vertices induce more than one edge in the A_4-structure. A graph is matrogenic if and only if no five of its vertices induce exactly two or three edges in the A_4-structure.*

A similar notion arises in the study of P_4-structures. The (q,t)-*graphs* were defined in [2] as those graphs on which no q vertices induce more than t copies of P_4; the P_4-free graphs are the $(4,0)$-graphs, and the P_4-*sparse* graphs [23] are the $(5,1)$-graphs. If we were to define the $[q,t]$-*graphs* as those in which no q vertices induced more than t edges in the A_4-structure, then the threshold graphs would be the $[4,0]$-graphs, and the matroidal graphs would be the $[5,1]$-graphs.

As a final motivation for our study of A_4-structure, we note that alternating 4-cycles and degree sequences are closely related to the *canonical decomposition* of a graph, defined by Tyshkevich in [51] (see also [49]). As we will show in this chapter, a graph is indecomposable with respect to the canonical decomposition if and only if its A_4-structure is connected.

In the remainder of this thesis, we provide some initial results on the A_4-structure of a graph. In Section 4.2 we examine the A_4-structures of cycles. We show that long cycles and their complements are the unique realizations of their respective A_4-structures; as a consequence, perfect graphs are recognizable from their A_4-structures. We also show that the A_4-structure in some sense determines the structure of matchings in a triangle-free graph. In Section 4.3 we show how the A_4-structure of a graph is related to its canonical decomposition, as defined by Tyshkevich [49, 51]. In Section 4.4 we show that A_4-structure, canonical decomposition, and vertex subsets known as *strict modules* satisfy analogues of several results on P_4-structure, modular decomposition, and graph modules. Finally, in Section 4.5 we discuss the problem of obtaining all realizations of a given A_4-structure, which leads us to characterize the *A_4-split graphs*, those graphs having the same A_4-structure as some split graph.

We conclude this section with some definitions. Given two graphs G and G' with A_4-structures H and H', respectively, we define a bijection $\varphi : V(G) \to V(G')$ to be an *A_4-isomorphism* from G to G' if it is a hypergraph isomorphism from H to H'. If an A_4-isomorphism exists from G to G', then we say that G and G' *have the same A_4-structure*, or that they are *A_4-isomorphic*.

4.2 A_4-structure and cycles

In this section we show that long cycles and their complements are characterized by their A_4-structures. As a consequence, perfect graphs may also be recognized from their A_4-structures. We conclude the section by showing how A_4-structure and matchings are related in triangle-free graphs.

In [13], Chvátal showed that odd cycles of length at least 5 and their complements are the only realizations of their respective P_4-structures, and he conjectured that two graphs with the same P_4-structure are either both perfect or both imperfect. Reed [47] proved this conjecture, now known as the Semistrong Perfect Graph Theorem since it implies the Perfect Graph Theorem of Lovász [34] and is in turn implied by the Strong Perfect Graph Theorem. This last result, proved much later by Chudnovsky et al. [12] states that a graph G is perfect if and only if no odd cycle of length at least 5 or its complement is an induced subgraph of G.

Motivated by the results of Chvátal and Reed, we show that for $n = 5$ and

$n \geq 7$, the cycle C_n and its complement are the only realizations of their A_4-structure. By the Strong Perfect Graph Theorem, it then follows that graphs with the same A_4-structures are either both perfect or both imperfect.

For any cycle C_n, the edges of the A_4-structure of C_n are the 4-sets consisting of two disjoint consecutive pairs of vertices in the cycle. We begin with some fundamental observations.

Observation 4.2. *If four vertices induce an alternating 4-cycle in a graph, then they also induce an alternating 4-cycle in the complement of the graph. Hence a graph and its complement have the same A_4-structure.*

Lemma 4.3. *In any graph, four vertices comprise an edge in the A_4-structure of the graph if and only if none of the vertices dominates or is isolated from the other three. Four vertices also comprise an edge in the A_4-structure if and only if no three of them form a clique or independent set in the graph.*

Proof. Recall that $2K_2$, P_4, and C_4 are the only 4-vertex graphs in which an alternating 4-cycle occurs. Of the eleven graphs on four vertices, these three graphs are the only graphs having neither a dominating nor an isolated vertex, and they are also the only graphs having no 3-clique or independent set of size 3. □

In the discussion that follows, suppose that C_n is A_4-isomorphic to a graph G. Denoting C_n by $[u_1, \ldots, u_n]$, we name the vertices of G as $v_1, \ldots, v_n$ so that u_i is mapped to v_i by a given A_4-isomorphism. Note that C_n is A_4-isomorphic to both G and $\overline{G}$ under this map. We will show that if v_1v_2 is an edge in G, then the A_4-isomorphism from C_n to G is in fact a graph isomorphism.

Let all addition and subtraction in the indices of vertices be done modulo n.

Lemma 4.4. *No triangle or independent set of size 3 in G can contain both v_i and v_{i+1} for some i.*

Proof. For any $i, j \in \{1, \dots, n\}$, from the description of the A_4-structure of C_n there is some edge of the A_4-structure containing v_i, v_{i+1}, and v_j; by Lemma 4.3, these vertices induce no triangle or $3K_1$ in G. $\square$

In what follows, define an *alternating path* $\langle u_1, \dots, u_j \rangle$ to be a configuration on distinct vertices $\{u_1, \dots, u_j\}$ such that pairs of consecutive vertices are alternately adjacent and non-adjacent. (Note that the usage of $\langle u_1, \dots, u_j \rangle$ to denote an alternating path is a departure from the notation of previous chapters, where it was used to describe paths. We will use this new notation throughout the rest of the thesis.) We denote the vertex set of an alternating path A by $V(A)$.

Lemma 4.5. *If $n \geq 7$, then the pairs v_i, v_{i+1} are either all adjacent or all non-adjacent in G.*

Proof. Suppose that the pairs v_i, v_{i+1} are not all adjacent and not all non-adjacent. There exists an index j such that exactly one of $v_{j-1}v_j$ and v_jv_{j+1} is an edge of G. Since exactly one of these pairs is an edge of $\overline{G}$, and G and $\overline{G}$ have the same A_4-structure, we may assume that $v_{j-1}v_{j+1} \notin E(G)$, and by symmetry we may assume that $v_{j-1}v_j \in E(G)$ and $v_jv_{j+1} \notin E(G)$. We illustrate the vertices $v_{j-3}, \dots, v_{j+2}$ of G in Figure 4.2. Since $n \geq 7$, these vertices are all distinct, and v_{j-3} and v_{j+2} are not consecutively-indexed vertices.

Let H be the A_4-structure of G. Since $E(H)$ contains both $\{v_{j-2}, v_{j-1}, v_j, v_{j+1}\}$ and $\{v_{j-1}, v_j, v_{j+1}, v_{j+2}\}$, Lemma 4.3 implies that $v_{j-2}v_{j+1}, v_{j+1}v_{j+2} \in E(G)$. By Lemma 4.4, we have $v_{j-2}v_{j+2} \notin E(G)$. Since $\{v_{j-3}, v_{j-2}, v_{j+1}, v_{j+2}\} \in E(H)$, we have $v_{j-3}v_{j+1} \notin E(G)$ by Lemma 4.3. By Lemma 4.4 we have $v_{j-3}v_j \in E(G)$. Since $\langle v_j, v_{j+1}, v_{j-2}, v_{j+2} \rangle$ is an alternating path in G but $\{v_{j-2}, v_j, v_{j+1}, v_{j+2}\} \notin E(H)$, we have $v_jv_{j+2} \notin E(G)$. However, we then have $\{v_{j-3}, v_j, v_{j+2}, v_{j+1}\} \in E(H)$, a contradiction. $\square$

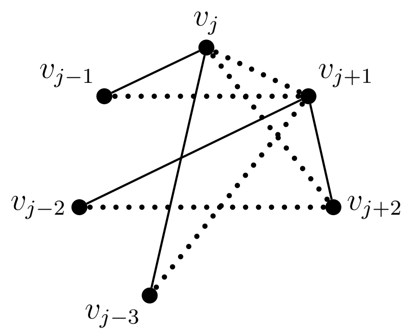


Figure 4.2: The subgraph of G from Lemma 4.5.

Assume now that $n \geq 7$. Since G and its complement have the same A_4-structure, we will also assume that $v_1v_2 \in E(G)$.

Lemma 4.6. *The graph G has no edges of the form v_iv_j where $|j - i| \neq 1$. Consequently, G is isomorphic to C_n.*

Proof. By Lemma 4.5, $[v_1, \ldots, v_n]$ is a spanning cycle of G. By Lemma 4.4, $v_iv_{i+2} \notin E(G)$ for all i. Suppose that G has a chord v_jv_k for vertices v_j and v_k at a distance of at least 3 on the cycle. By Lemma 4.4, $v_jv_{k-1}, v_jv_{k+1} \notin E(G)$. It follows that $[v_j, v_k : v_{k-2}, v_{k-1}]$ and $[v_j, v_k : v_{k+2}, v_{k+1}]$ are alternating 4-cycles in G. Since $n \geq 7$, either v_{k-2} or v_{k+2} is not consecutive to v_j, which contradicts the description of the A_4-structure of C_n. Thus G has no chords and hence is isomorphic to C_n. $\square$

Recall that C_5 is the only graph with five vertices whose A_4-structure has more than three edges. The results above imply the following.

Theorem 4.7. *If $n = 5$ or $n \geq 7$, then C_n and its complement are the only graphs having their A_4-structure.*

Corollary 4.8. *If two graphs have the same A_4-structure, then they are either both perfect or both imperfect.*

Proof. Suppose that G and G' have the same A_4-structure, and let $\varphi : V(G) \to V(G')$ be an A_4-isomorphism. Let n be an odd integer such that $n \geq 5$. By Theorem 4.7, G induces C_n or $\overline{C_n}$ on a vertex subset S if and only if G' induces C_n or $\overline{C_n}$ on $\varphi(S)$. The Strong Perfect Graph Theorem then implies the result. □

The conclusion of Theorem 4.7 does not hold when $n = 6$; the graph C_6 shares its A_4-structure with G' and $\overline{G'}$, where G' is any graph obtained by deleting up to three pairwise non-incident edges from $K_{3,3}$. Note also that Theorem 4.7 applies to long cycles of both parities, whereas Chvátal's analogous result for P_4-structure deals only with odd cycles.

We conclude our discussion of cycles and A_4-structure by presenting a result on matchings in triangle-free graphs. A graph G has a *perfect matching* if it has a matching of size $\frac{1}{2}|V(G)|$.

Lemma 4.9. *If G is a 6-vertex triangle-free graph whose vertex set can be partitioned into three pairs of vertices such that the union of any two of these pairs is an edge in the A_4-structure of G, then G has a perfect matching.*

Proof. Let H be the A_4-structure of G, and let A, B, and C denote the vertex pairs described, so that $V(G) = A \cup B \cup C$ and $A \cup B, A \cup C, B \cup C \in E(H)$. If the vertices in each of A, B, and C induce an edge in G, then G has a perfect matching. If not, then we may assume without loss of generality that $a_1a_2 \notin E(G)$, where $A = \{a_1, a_2\}$. Since $A \cup B \in E(H)$, vertices a_1 and a_2 belong to non-incident edges a_1b_1, a_2b_2 in $G[A \cup B]$. Similarly, there exist non-incident edges a_1c_1, a_2c_2 in $G[A \cup C]$. Since G is triangle-free, $b_1c_1, b_2c_2 \notin E(G)$. However, $B \cup C \in E(H)$, so $B \cup C$ induces two non-incident edges. It follows that G has a spanning cycle and hence a perfect matching. □

For graphs G and G', we say that a bijection $\varphi : V(G) \to V(G')$ *preserves matchings* if a set S is the vertex set of a matching of size at least 2 in G if and only if $\varphi(S)$ is the vertex set of a matching in G'.

Theorem 4.10. *Let G and G' be triangle-free graphs, and let $\varphi : V(G) \to V(G')$ be a bijection. The map φ is an A_4-isomorphism if and only if it preserves matchings.*

Proof. Suppose that φ preserves matchings. In a triangle-free graph G, the four vertices spanned by a matching of size 2 contain no 3-clique or independent set of size 3, so by Lemma 4.3 these vertices form an edge in the A_4-structure of G. Conversely, the three 4-vertex graphs $2K_2$, P_4, and C_4 that have an alternating 4-cycle all have perfect matchings. Thus a vertex subset S in G induces an alternating 4-cycle if and only if $\varphi(S)$ induces an alternating 4-cycle in G'; hence φ is an A_4-isomorphism.

Suppose instead that $\varphi : V(G) \to V(G')$ is an A_4-isomorphism, and let S be the vertex set of some matching in G of size at least 2. We may partition the edges of the matching on S into pairs and triples of edges; let $S_1, S_2, \ldots, S_j$ be the vertex sets of these edge sets. By the previous paragraph and Lemma 4.9, the sets $\varphi(S_i)$ are the vertex sets of disjoint matchings in G'. The union of these matchings is a matching on $\varphi(S)$, so φ preserves matchings. □

4.3 Canonical decomposition and A_4-structure

In this section we describe the relationship that the A_4-structure of a graph has with its canonical decomposition, as defined by Tyshkevich [49, 51].

A *splitted graph* is a triple (G, A, B) such that G is a split graph whose vertices partition into an independent set A and a clique B. Two splitted graphs (G, A, B) and (G', A', B') are *isomorphic* if there exists a graph isomorphism $\varphi : V(G) \to V(G')$ such that $\varphi(A) = A'$. Given a splitted graph (G, A, B) and a graph H on disjoint vertex sets, we define the *composition* of (G, A, B) and H to be the

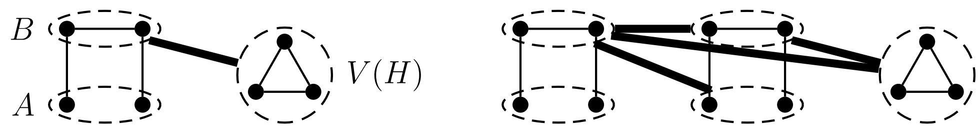


Figure 4.3: The compositions $(G, A, B) \circ H$ and $(G, A, B) \circ (G, A, B) \circ H$.

graph $(G, A, B) \circ H$ formed by adding to $G + H$ all edges uv such that $u \in B$ and $v \in V(H)$. For example, when $H = K_3$ and $G = P_4$, with A the set of endpoints and B the set of midpoints of G, the composition $(G, A, B) \circ K_3$ is the graph on the left in Figure 4.3 (here and in the future, heavy lines joining sets of vertices imply that all edges joining vertices from one set to the other are present). On the right we show $(G, A, B) \circ ((G, A, B) \circ K_3)$. The operation $\circ$ is associative, so in the future we will omit grouping parentheses when performing multiple compositions. Observe that in a composition $(G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0$, each vertex in B_i is adjacent to every vertex in $\bigcup_{j<i} V(G_j)$, each vertex in A_i is adjacent to none of the vertices in $\bigcup_{j<i} V(G_j)$, and only the rightmost graph in the composition can fail to be a split graph.

A graph is *decomposable* if it can be written as a composition $(G, A, B) \circ H$, where G and H both have at least one vertex. Otherwise, it is *indecomposable*. Tyshkevich showed the following:

Theorem 4.11 (Tyshkevich [49]). (i) *Every graph G can be expressed as a composition*

$$G = (G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0 \qquad (*)$$

of indecomposable components. Here the (G_i, A_i, B_i) are indecomposable splitted graphs and G_0 is an indecomposable graph. (If G is indecomposable, then $k = 0$; that is, there are no splitted components in $(*)$).

(ii) *Graphs G and G' expressed as* $(*)$ *and*

$$G' = (G'_\ell, A'_\ell, B'_\ell) \circ \cdots \circ (G'_1, A'_1, B'_1) \circ G'_0$$

are isomorphic if and only if the following conditions hold:

(1) $G_0 \cong G'_0$,

(2) $k = \ell$,

(3) $(G_i, A_i, B_i) \cong (G'_i, A'_i, B'_i)$ *for* $1 \leq i \leq k$.

Theorem 4.11 implies that there is only one such composition of a graph G into indecomposable components, up to isomorphism of the components. Therefore, we call it the *canonical decomposition*.

The following result provides a characterization of indecomposable graphs in terms of their A_4-structures.

Theorem 4.12. *A graph is indecomposable with respect to canonical decomposition if and only if its A_4-structure is connected. Hence, the vertex sets of the G_i in the canonical decomposition*

$$G = (G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0$$

are exactly the vertex sets of the components in the A_4-structure of G.

The proof is lengthy, so we first prove several preliminary results. Given an alternating 4-cycle $C = [a, b : c, d]$, let $V(C)$ denote the set $\{a, b, c, d\}$.

Observation 4.13. *If a graph G has more than one vertex and has canonical decomposition $(G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0$, then G has an isolated vertex or dominating vertex if and only if $k \geq 1$ and G_k has exactly one vertex. The vertex is dominating in G if $A_k = \emptyset$ and is isolated in G if $B_k = \emptyset$.*

Observation 4.14. *If* $G = (G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0$, *then* $\overline{G} = (\overline{G_k}, B_k, A_k) \circ \cdots \circ (\overline{G_1}, B_1, A_1) \circ \overline{G_0}$.

Proposition 4.15. *If G is an indecomposable graph with more than one vertex, then every vertex of G belongs to an alternating 4-cycle in G.*

Proof. We prove the contrapositive. Suppose that some vertex v in G belongs to no alternating 4-cycle. If v is a dominating or isolated vertex, then G is decomposable by Observation 4.13, so we may assume that v is neither. Let $V_1 = N(v)$ and $V_2 = V(G) \setminus N[v]$.

If V_1 is not a clique, then there exist $u, w \in V_1$ such that $uw \notin E(G)$. For $a \in V_2$, since neither $[v, w : u, a]$ nor $[v, u : w, a]$ is an alternating 4-cycle (both contain v), a is adjacent to neither u nor w. Hence if $A = \{x \in V_1 : N_G(x) \cap V_2 \neq \emptyset\}$, then each vertex in A dominates V_1, which makes A a clique. Furthermore, V_2 is independent, since if a and b were adjacent vertices in V_2, then $[v, u : a, b]$ would be an alternating 4-cycle containing v. Letting $B = V_1 \setminus A$, we obtain a decomposition $G = (G', V_2, A) \circ (K_1, \emptyset, \{v\}) \circ G[B]$, where $G' = G[V_2 \cup A]$. Since G has more than one vertex, at least one of V_2, A, B is nonempty, so G is decomposable. Hence we may assume that V_1 is a clique in G.

We note that the complement of an alternating 4-cycle is an alternating 4-cycle, so v belongs to no alternating 4-cycle in $\overline{G}$. Since $N_{\overline{G}}(v) = V_2$ and $V(\overline{G}) \setminus N_{\overline{G}}[v] = V_1$, the preceding argument shows that either $\overline{G}$ (and hence G, by Observation 4.14) is decomposable or V_2 is a clique in $\overline{G}$ and hence an independent set in G. We assume the latter.

With V_1 a clique and V_2 an independent set, we have $G = (G', V_2, V_1) \circ G[\{v\}]$, where $G' = G[V_2 \cup V_1]$. Hence in all cases G is decomposable. $\square$

Given alternating 4-cycles $A = [a, b : c, d]$ and $B = [e, f : g, h]$ in G, we define the relation $A \to B$ to mean that $G[V(A)] \cong P_4$, the midpoints of $G[V(A)]$

dominate $V(B)$, and the endpoints of $G[V(A)]$ are nonadjacent to each vertex in $V(B)$.

Lemma 4.16. *If A and B are disjoint alternating 4-cycles in a graph G such that no alternating 4-cycle in G intersects both A and B, then either $A \to B$ or $B \to A$.*

Proof. Let $A = [a, b : c, d]$ and $B = [e, f : g, h]$. Since $\{a, b, e, f\}$ is not the vertex set of an alternating 4-cycle in G, by Lemma 4.3 one of these four vertices dominates the other three; suppose that a is this vertex. Since neither $[a, f : g, h]$ nor $[a, e : h, g]$ is an alternating 4-cycle in G, we have $ag, ah \in E(G)$. Thus a dominates $V(B)$. It follows that d has no neighbor v in $V(B)$, for otherwise $[a, u : v, d]$ would be an alternating 4-cycle, where u is the non-neighbor of v in B. Making the same argument starting with $\{c, d, g, h\}$ now implies that c dominates B and b has no neighbor in $V(B)$.

Finally, note that $bd \notin E(G)$, since otherwise $[b, d : e, f]$ would be an alternating 4-cycle, and $ac \in E(G)$, since otherwise $[a, e : h, c]$ would be an alternating 4-cycle. We conclude that $G[V(A)] \cong P_4$, with midpoints a, c dominating B and endpoints b, d adjacent to no vertex of B. Thus $A \to B$.

The same conclusion holds by a symmetric argument if b dominates $\{a, e, f\}$. If instead e or f dominates the other three vertices of $\{a, b, e, f\}$, then we arrive similarly at $B \to A$. $\square$

This last result shows, incidentally, that if two vertices each belong to an induced $2K_2$ or C_4, then they have distance at most 3 in the A_4-structure of the graph, since some edge of the A_4-structure must intersect these edges containing them. We also have the following result.

Corollary 4.17. *Let G be a graph, and let H be the A_4-structure of G. If A and B are alternating 4-cycles in G such that $V(A)$ and $V(B)$ are contained in* *distinct components of H, then $A \to B$ or $B \to A$.*

Lemma 4.18. *If A, B, and C are alternating 4-cycles in a graph G such that $A \to B$ and $V(A) \cap V(C)$ is nonempty, then $B \nrightarrow C$.*

Proof. If $B \to C$, then the midpoints of the path induced by B dominate C, and the endpoints have no neighbors in C. Hence no vertex in C can dominate or be independent of B. This requires $V(A) \cap V(C) = \emptyset$. □

Proposition 4.19. *Let G be a graph, and let H be the A_4-structure of G. Let Q_1 and Q_2 be distinct components of H, and let A and B be alternating 4-cycles in G such that $V(A) \subseteq V(Q_1)$ and $V(B) \subseteq V(Q_2)$. If $A \to B$, then $C \to D$ for any alternating 4-cycles C and D in G such that $V(C) \subseteq V(Q_1)$ and $V(D) \subseteq V(Q_2)$.*

Proof. Since $V(B)$ and $V(D)$ both lie in $V(Q_2)$, there are alternating 4-cycles $R_0, R_1, \ldots, R_k$ such that $B = R_0$, $D = R_k$, and $V(R_{i-1}) \cap V(R_i) \neq \emptyset$ for $1 \leq i \leq k$. By Corollary 4.17, $A \to R_i$ or $R_i \to A$ for each i. If $R_1 \to A$, then Lemma 4.18 implies $A \nrightarrow B$, which is false. Hence $A \to R_1$. Iterating the argument yields $A \to R_i$ for all $i \in \{1, \ldots, k\}$. In particular, $A \to D$.

Similarly, since $V(A)$ and $V(C)$ lie in $V(Q_1)$, there are alternating 4-cycles $S_0, \ldots, S_\ell$ with $A = S_0$, $C = S_\ell$, and $V(S_{i-1}) \cap V(S_i) \neq \emptyset$ for $i = 1, \ldots, \ell$. Corollary 4.17 implies that $S_i \to D$ or $D \to S_i$ for each i. Since $A \to D$, Lemma 4.18 yields $S_1 \to D$. Again iterating the argument, we conclude that $C \to D$. □

In the following, let H be the A_4-structure of a graph G. We define a relation on the components of H. Given components Q_1, Q_2 of H, we write $Q_1 \to Q_2$ if Q_1 contains an alternating 4-cycle A and Q_2 contains an alternating 4-cycle B such that $A \to B$. By Proposition 4.19, $Q_1 \to Q_2$ implies $Q_2 \nrightarrow Q_1$.

Assume now that G is indecomposable in the canonical decomposition. Proposition 4.15 implies that each component of H contains at least one alternating

4-cycle, so for any two components Q_1, Q_2 of H, either $Q_1 \to Q_2$ or $Q_2 \to Q_1$. We may now define a tournament T whose vertices are the components of H, with edges oriented according to the relation $\to$ on the components of H.

Lemma 4.20. *The tournament T is acyclic.*

Proof. If T contains a cycle, then T contains a cyclic triangle with vertices Q_1, Q_2, Q_3 in order. By Proposition 4.19, it follows that there are alternating 4-cycles A_1, A_2, A_3 with $V(A_i) \subseteq V(Q_i)$ for $i \in \{1, 2, 3\}$, such that $A_1 \to A_2$, $A_2 \to A_3$, and $A_3 \to A_1$. In particular, $G[V(A_1)] \cong G[V(A_2)] \cong G[V(A_3)] \cong P_4$.

Let a denote a vertex of degree 2 in $G[V(A_1)]$, let b denote a vertex of degree 1 in $G[V(A_2)]$, let c denote a vertex of degree 2 in $G[V(A_3)]$, and let d denote the vertex of degree 1 in $G[V(A_3)]$ adjacent to c. The adjacencies implied by the $\to$ relation on the A_i imply that $[a, b : c, d]$ is an alternating 4-cycle in G. This contradicts that Q_1, Q_2, and Q_3 are distinct components in H. □

Proposition 4.21. *Any two vertices that lie on an alternating 4-cycle in G belong to the same component in the canonical decomposition of G.*

Proof. Let $(G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0$ be the canonical decomposition of G. Suppose that some alternating 4-cycle $[u, v : w, x]$ has vertices in more than one G_i. Let j be the largest index such that G_j contains some vertex of the alternating 4-cycle, and assume without loss of generality that u belongs to $V(G_j)$. Suppose first that $u \in B_j$. Since x is not adjacent to u, x cannot belong to B_j or to $V(G_i)$ for $i < j$; thus $x \in A_j$. Since w is adjacent to x, vertex w cannot belong to A_j or to $V(G_i)$ for $i < j$; hence $w \in B_j$. Repeating the argument for x now yields $v \in A_j$. Thus $\{u, v, w, x\} \in V(G_j)$. A similar result follows if we instead start with $u \in A_j$. □

We are now ready to prove our main result:

Proof of Theorem 4.12. Let G be an arbitrary graph, and let H be its A_4-structure.

Suppose first that H is connected. For $u, v \in V(G)$, there exist edges $E_0, \ldots, E_k$ of H such that $u \in E_0$ and $v \in E_k$, and $E_i \cap E_{i-1} \neq \emptyset$ for $1 \leq i \leq k$. Applying Proposition 4.21 to vertices in the sets $E_0, \ldots, E_k$ in turn, we find that u and v belong to the same component in the canonical decomposition of G. Thus G is indecomposable.

Suppose instead that H is disconnected. If G is not decomposable, then Proposition 4.15 implies that each component of H contains at least one alternating 4-cycle, so the relation $\rightarrow$ is defined on the components of H, and the acyclic tournament T described above exists. Let Q be the component of H that is the source vertex of T. Every alternating 4-cycle in $G[V(Q)]$ corresponds to an induced P_4 in G whose midpoints dominate every vertex not in $V(Q)$, and whose endpoints only have neighbors in $V(Q)$. These adjacency requirements ensure that no vertex in $V(Q)$ is both a midpoint of some induced P_4 and an endpoint of another. Thus we may partition $V(Q)$ into sets A and B, where A and B denote the set of all endpoints and the set of all midpoints of induced P_4's in $G[V(Q)]$, respectively. Let $[a, b : c, d]$ be an alternating 4-cycle of G whose vertices belong to $V(G) \setminus V(Q)$. Further let s and t be any vertices in A, and let u and v be any vertices of B. If s and t are adjacent, then $[s, t : a, b]$ is an alternating 4-cycle in G, which contradicts the assumption that $a, b \notin V(Q)$. Similarly, if u, v are non-adjacent in G, then G contains the alternating 4-cycle $[b, u : v, c]$, again a contradiction. We conclude that B is a clique and A is an independent set in G. Hence $G = (G', A, B) \circ G[V(G) \setminus V(Q)]$, where $G' = G[A \cup B]$, and G is decomposable.

Having shown that G is indecomposable if and only if H is connected, it follows immediately that the components of H partition the set $V(G)$ into exactly the same subsets that the components in the canonical decomposition do. □

Theorem 4.12 provides a connection between the A_4-structure of a graph and its degree sequence. This is not surprising, since alternating 4-cycles play an important role in realizations of degree sequences. Tyshkevich [49, 51] provided a characterization of indecomposable graphs in terms of their degree sequences and showed how the canonical decomposition of a graph corresponds precisely to a decomposition of the degree sequence of the graph. In particular, she showed the following.

Proposition 4.22 (Tyshkevich [49, 51])**.** *For every graph G, the degree sequence of G uniquely determines the number of indecomposable components present in the canonical decomposition of G and how many vertices each indecomposable component contains.*

It follows immediately that graphs with the same degree sequence have A_4-structures with some features the same.

Corollary 4.23. *If G and G' are graphs with the same degree sequence, then G and G' have the same number and sizes of components in their A_4-structures.*

4.4 A_4-structure and modules

Based on the results of the previous section, we show in this section how A_4-structures and the canonical decomposition have a relationship much like that of P_4-structures and other graph decompositions. We begin with some facts about modules and the P_4-structure of a graph. Our presentation follows that of Hougardy [28].

A *module* in a graph G is a set S of vertices such that every vertex outside S is either adjacent to all vertices of S or to no vertex of S. A module S is *trivial* if

$|S| = 1$ or $S = V(G)$, and a graph is *prime* if it has no nontrivial modules. The modules in a graph are related to the vertex sets inducing P_4 via the following result.

Lemma 4.24 (Seinsche [48])**.** *The following hold for every graph G.*

(i) *The vertex set of an induced P_4 in G and a module in G can only intersect in zero, one, or four vertices.*

(ii) *G is P_4-free if and only if every induced subgraph with at least three vertices contains a nontrivial module.*

The *modular decomposition* of a graph recursively partitions its vertex set into modules via the following result.

Theorem 4.25 (Gallai [18])**.** *Let G be a graph with at least two vertices. Exactly one of the following conditions holds.*

(i) *G is disconnected.*

(ii) *$\overline{G}$ is disconnected.*

(iii) *There exists a subset Y of $V(G)$, where $|Y| \geq 4$, and a unique partition $V_1, \ldots, V_k$ of $V(G)$ such that Y induces a maximal prime subgraph in G and every V_i is a module with $|V_i \cap Y| = 1$.*

Jamison and Olariu [29] provided a refinement of the modular decomposition called the *primeval decomposition*, which makes use of the P_4-structure of the graph. A graph G is *p-connected* if for every partition of its vertex set into two nonempty disjoint sets, there exists an edge in the P_4-structure that intersects both sets. A maximal p-connected induced subgraph of G is a *p-component*. A p-connected graph G is *separable* if its vertex set can be partitioned into two nonempty disjoint sets such that each P_4 in G that is not completely contained within one of the sets has its endpoints in one set and its midpoints in the other set. The primeval decomposition of a graph partitions its vertex set into modules via applications of the following theorem.

Theorem 4.26 (Jamison–Olariu [29]). *For a graph G, exactly one of the following conditions holds.*

(i) *G is disconnected.*

(ii) *$\overline{G}$ is disconnected.*

(iii) *G is p-connected.*

(iv) *There is a unique proper separable p-component Q of G with a partition Q_1, Q_2 of $V(Q)$ such that every vertex not in $V(Q)$ is adjacent to all vertices in Q_1 and not adjacent to any vertex in Q_2.*

We now define a type of module that will play for A_4-structures much the same role that ordinary modules do for P_4-structures. Observe that a vertex subset S in a graph G is a module if and only if there is no triple v_1, v_2, v_3 in G such that $v_1, v_3 \in S$, $v_2 \notin S$, $v_1v_2 \in E(G)$, and $v_2v_3 \notin E(G)$. We generalize this forbidden configuration: an alternating path $\langle v_1, \ldots, v_p \rangle$ is *S-terminal* if $p \geq 3$ and $S \cap \{v_1, \ldots, v_p\} = \{v_1, v_p\}$. We allow the possibility that the $v_1 = v_p$, but otherwise the vertices are distinct. Define a *strict module* to be a vertex subset S of $V(G)$ such that G has no S-terminal alternating path. Strict modules are clearly modules. We show next that the condition for strict modules can be simplified. The length of an alternating path $\langle v_1, \ldots, v_p \rangle$ is defined to be $p - 1$.

Proposition 4.27. *A vertex subset S is a strict module of G if and only if G has no S-terminal alternating paths of length 2 or 3.*

Proof. If S is a strict module, then by definition G contains no S-terminal alternating paths of lengths 2 or 3. If S is not a strict module, then there is an S-terminal alternating path in G; let $\langle v_1, \ldots, v_p \rangle$ be a shortest one. If $p \geq 5$, then consider v_3. Whether v_3 is adjacent to v_1 or not, the alternating nature of the original allows a new path to be continued from v_3 to v_2 or v_4. That is, $\langle v_1, v_3, v_2, v_1 \rangle$ or $\langle v_1, v_3, v_4, \ldots, v_p \rangle$ is a shorter S-terminal alternating path. Thus $p \leq 4$. □

As with modules, let us call a strict module S in a graph G *trivial* if $S = V(G)$. Note, however, that single vertices in G need not comprise strict modules. Proposition 4.29 below provides an analogue to Lemma 4.24. First, we recall that the threshold graphs are those that have no alternating 4-cycles. The threshold graphs have the following two characterizations.

Theorem 4.28 (Chvátal–Hammer [14]). *The following are equivalent and characterize the threshold graphs G.*

(i) *G is $\{2K_2, C_4, P_4\}$-free.*

(ii) *G can be constructed by starting with a single vertex and iteratively adding either an isolated vertex or a dominating vertex to the graph.*

Proposition 4.29. *The following hold for every graph G.*

(i) *Every alternating 4-cycle in G and strict module in G intersect in zero or four vertices.*

(ii) *G contains no alternating 4-cycles if and only if every induced subgraph with at least two vertices contains a nontrivial strict module.*

Proof. (i) One easily checks that if a vertex subset S in G contains exactly one, two, or three vertices of an alternating 4-cycle in G, then some subset of the vertices of the alternating 4-cycle comprise an S-terminal alternating path, so S is not a strict module.

(ii) If every induced subgraph with more than one vertex has a strict module, then G is $\{2K_2, C_4, P_4\}$-free, since none of these graphs has a strict module. By Theorem 4.28, G has no alternating 4-cycle. Conversely, if G has no alternating 4-cycles, then G is a threshold graph. Hence every induced subgraph H with at least two vertices has a vertex u that is either dominating or isolated in H. Now $V(H) - u$ is a strict module in H. $\square$

We shall later prove an analogue of Theorem 4.26 related to the A_4-structure of a graph. First, we examine the structure of a graph in terms of its strict modules.

Proposition 4.30. *Let G be a graph with strict module S. If A and B are the sets of all vertices in $V(G)-S$ that are adjacent to none of S or to all of S, respectively, then A is an independent set and B is a clique in G. Hence $G = (G_1, A, B) \circ G[S]$, where $G_1 = G[A \cup B]$.*

Proof. If two vertices in A are non-adjacent, or if two vertices of B are adjacent, then these vertices form the midpoints of a (possibly closed) S-terminal alternating path of length 3, which cannot happen when S is a strict module. $\square$

In any composition $(G, A, B) \circ H$, the vertex set of H is a strict module. We thus conclude the following.

Corollary 4.31. *A graph G is indecomposable with respect to canonical decomposition if and only if it has no nontrivial strict module.*

Corollary 4.31 shows that in the study of strict modules, the indecomposable graphs play a role analogous to that of the prime graphs for (ordinary) modules.

We turn our attention now to presenting an analogue of Theorem 4.26 in terms of A_4-structures and the canonical decomposition. Define an A_4-*component* of a

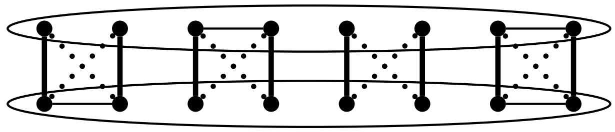


Figure 4.4: Alternating 4-cycles in an A_4-separable graph.

graph G to be an induced subgraph of G whose vertex set is the vertex set of some component of the A_4-structure of G. Theorem 4.12 shows that the A_4-components of G are precisely the components of the canonical decomposition of G.

Define a graph to be A_4-*separable* if there exists a partition of its vertex set into two subsets V and W such that every induced P_4 has its endpoints in one of V, W and its midpoints in the other, every induced $2K_2$ has one pair of nonadjacent vertices in V and the other two vertices in W, and every induced C_4 has two adjacent vertices in V and the other two in W. In other words, a graph is A_4-separable if every 4-vertex induced subgraph having an alternating 4-cycle has an alternating 4-cycle whose vertices alternate between V and W, as shown in Figure 4.4.

Note that every split graph S is A_4-separable; letting V and W partition $V(S)$ into an independent set and a clique, respectively, this claim follows immediately from the following results.

Proposition 4.32 (Földes–Hammer [16]). *A graph is split if and only if it is* $\{2K_2, C_4, C_5\}$*-free.*

Observation 4.33. *In any partition of the vertex set of a split graph S into a clique Q and an independent set I, every induced path on four vertices has its midpoints in Q and its endpoints in I.*

We will say more about A_4-separable graphs in the following section. We conclude this section with the analogue for A_4-structure of Theorem 4.26.

Proposition 4.34. *For any graph G having more than one vertex and having canonical decomposition*

$$G = (G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0,$$

exactly one of the following is true:

(i) *G has an isolated vertex.*

(ii) *$\overline{G}$ has an isolated vertex.*

(iii) *G has a connected A_4-structure.*

(iv) *$k \geq 1$, and G_k is the unique A_4-separable A_4-component Q of G having a partition of $V(Q)$ into nonempty subsets Q_1, Q_2 such that every vertex not in $V(Q)$ is adjacent in G to no vertices in Q_1 and to all vertices in Q_2; here $Q_1 = A_k$ and $Q_2 = B_k$.*

Proof. We have observed already that no two of (i), (ii), (iii) can simultaneously hold. If (iv) holds, then since G_k has at least two vertices, it follows from Observation 4.13 and Theorem 4.12 that none of (i), (ii), or (iii) holds. Hence at most one of these statements holds for G.

If none of (i), (ii), or (iii) holds, then $k \geq 1$. By the definition of the canonical decomposition, G_k induces a split graph with independent set A_k and clique B_k, and it follows from Observation 4.13 that A_k and B_k are nonempty. Since split graphs are A_4-separable, we see that G_k is an A_4-component of G having the properties described in (iv). That G_k is the only such A_4-component follows immediately from the definition of canonical decomposition when A_k and B_k are nonempty. □

4.5 A_4-split graphs

In this section we characterize the A_4-*split graphs*, those having the same A_4-structure as some split graph. As motivation, we show that this problem arises in the problem of constructing all graphs having a given A_4-structure.

Example 4.35. Graphs with the same A_4-structures. As shown in Proposition 4.21, any alternating 4-cycle is contained within a single component of the canonical decomposition. It follows that permuting the indecomposable components in a canonical decomposition does not change the A_4-structure of a graph.

By Theorem 4.12 and Proposition 4.21, each component of the A_4-structure of a graph is uniquely determined by the indecomposable component of the canonical decomposition on the same vertex set. If we replace an indecomposable component of the canonical decomposition with another subgraph having the same A_4-structure, the resulting graph will have the same A_4-structure as the original.

To illustrate these two A_4-structure-preserving operations, let G_2 be a graph consisting of a single vertex u, let G_1 consist of the single vertex v, and let $G_0 = K_2 + P_3$. Given the graph G with canonical decomposition $(G_2, \emptyset, \{u\}) \circ (G_1, \{v\}, \emptyset) \circ G_0$, let H be the graph formed by transposing the first two of the indecomposable components in the canonical decomposition; that is, $H = (G_1, \{v\}, \emptyset) \circ (G_2, \emptyset, \{u\}) \circ G_0$. Let G_0' be the 5-vertex graph with degree sequence $(3, 2, 1, 1, 1)$; note that G_0 and G_0' have the same A_4-structure. Let I be the graph formed from G by replacing the indecomposable component G_0 with G_0'; this is, $I = (G_2, \emptyset, \{u\}) \circ (G_1, \{v\}, \emptyset) \circ G_0'$. Graphs G, H, and I are illustrated in Figure 4.5. Though the graphs are pairwise nonisomorphic, all have the same A_4-structure.

For a graph G with canonical decomposition $(G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0$, we refer to the subgraph G_0 of G as the *core* of G. Note that the indecomposable

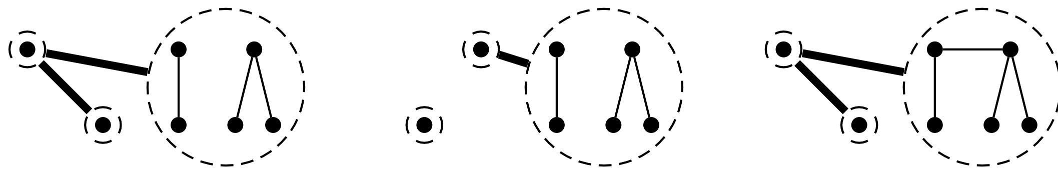


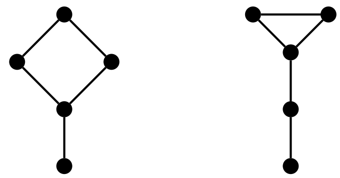
Figure 4.5: Different graphs with the same A_4-structure.

components of G other than the core are all split graphs. In order to generate other graphs having the same A_4-structure as G, we may wish to permute the indecomposable components of G under the canonical decomposition. However, if the core G_0 is not split, then we may not move the vertices of G_0 to a different position in the canonical decomposition unless we first replace G_0 by a split subgraph G_0' having the same A_4-structure. In order to determine if this is possible, we need a characterization of those graphs having the same A_4-structure as a split graph, i.e., the A_4-split graphs.

We preface our characterization with a few definitions. An A_4-structure H is *balanced* if there is a partition of $V(H)$ into two sets V_1 and V_2 such that every edge e of H has two vertices in V_1 and two vertices in V_2; the sets V_1 and V_2 then form a *balancing partition* of $V(H)$. A graph is A_4-*balanced* if its A_4-structure is balanced.

Given a balanced A_4-structure with a fixed balancing partition V_1, V_2 and a vertex v belonging to V_i, the v-*restriction* of H is the graph on V_{3-i} where two vertices are adjacent if and only if they are the two vertices in V_{3-i} of some edge of H containing v. A balanced A_4-structure H has the *bipartite restriction property* if there is a balancing partition of $V(H)$ such that for every vertex $v \in V(H)$ the v-restriction of H is bipartite.

The *k-pan* is the graph obtained by attaching a pendant vertex to a vertex of a k-cycle; the *co-k-pan* is its complement. The 4-pan and co-4-pan are illustrated



(a) (b)

Figure 4.6: (a) The 4-pan; (b) the co-4-pan.

in Figure 4.6.

Finally, recall that a graph G is A_4-*separable* if there is a partition of $V(G)$ into two sets V_1 and V_2 such that every 4-vertex induced subgraph having an alternating 4-cycle has an alternating 4-cycle whose vertices alternate between V_1 and V_2. The partition V_1, V_2 is an A_4-*separating* partition of $V(G)$.

Theorem 4.36. *For a graph G with core G_0 and A_4-structure H, the following statements are equivalent.*

(a) *G is A_4-split.*

(b) *H is balanced and has the bipartite restriction property.*

(c) *G and $\overline{G}$ are both $\{C_5, P_5, K_2 + K_3, \text{co-4-pan}, K_2 + P_4, K_2 + C_4, 2K_2 \vee 2K_1\}$-free.*

(d) *G is split, or G_0 or $\overline{G_0}$ is a disjoint union of stars.*

(e) *G is A_4-separable.*

Proof. We first show that (a) implies (b). Let G be an A_4-split graph, let H be its A_4-structure, and let G' be a split graph whose A_4-structure also is H. If V_1, V_2 is a partition of $V(G')$ into a clique and an independent set, then Proposition 4.32 and Observation 4.33 imply that V_1, V_2 is a balancing partition of $V(G)$; thus H is balanced. Furthermore, in an arbitrary copy of P_4 in G' with vertices $a_1, a_2 \in V_1$

and $b_1, b_2 \in V_2$, each a_i has exactly one neighbor in $\{b_1, b_2\}$, and each b_i has exactly one neighbor in $\{a_1, a_2\}$. Hence if v is a vertex of G', and B is the v-restriction of H, then for any edge xy in B, v is adjacent in G' to exactly one of x and y. It follows that giving the neighbors and nonneighbors of v in $V(B)$ opposite colors yields a proper 2-coloring of B. Thus B is bipartite. We conclude that H has the bipartite restriction property.

The property of being A_4-balanced is preserved under graph complementation and taking induced subgraphs, as is the property of having an A_4-structure with the bipartite restriction property. To show that (b) implies (c), it thus suffices to show that each of the graphs listed in (c) is either not A_4-balanced or does not have an A_4-structure with the bipartite restriction property. One verifies easily that C_5 is not A_4-balanced. The graphs P_5, $K_2 + K_3$, and the co-4-pan each have the same A_4-structure H^*. In H^*, the unique balancing partition has two vertices in one set and three vertices in the other, and the v-restriction of H^* for a vertex v in the set of size two is isomorphic to K_3, so H^* does not have the bipartite restriction property. The A_4-structures of $K_2 + P_4$, $K_2 + C_4$, and $2K_2 \vee 2K_1$ each have a unique balancing partition and a vertex v in each set of the partition such that the v-restriction of the A_4-structure is isomorphic to K_3.

We next show that (c) implies (d). Suppose that neither G nor $\overline{G}$ contains any of the graphs listed in (c) as an induced subgraph, and further suppose that G is not split. It follows that the indecomposable core G_0 of G is not split. Since G_0 is C_5-free, Proposition 4.32 implies that G_0 induces $2K_2$ or C_4. Suppose first that G_0 induces $2K_2$ on vertices a, b, c, d, with edges ab and cd. Since G is $\{K_2 + K_3, P_5, \overline{P}\}$-free, every other vertex in G_0 is adjacent to 0, 1, or 4 vertices in $\{a, b, c, d\}$. Let X denote the set of vertices adjacent to all four vertices in $\{a, b, c, d\}$, and let Y be the set of vertices adjacent to none of a, b, c, and d. Let A, B, C, and D denote the sets of vertices from $V(G_0) - \{a, b, c, d\}$ whose

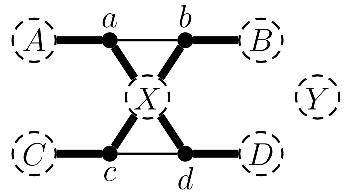


Figure 4.7: The graph G from Theorem 4.36.

neighborhoods intersect $\{a, b, c, d\}$ in $\{a\}$, $\{b\}$, $\{c\}$, and $\{d\}$, respectively. These vertices and sets are illustrated in Figure 4.7.

Since G_0 is $(2K_2 \vee 2K_1)$-free, X must be a clique. Suppose that X is nonempty, and let x be an arbitrary vertex in X. Since G_0 is co-4-pan-free, A, B, C, and D must all be empty. Let Y' be the set of vertices in Y having a neighbor in Y, and let $Y'' = Y - Y'$. Note that Y'' is an independent set. Any two adjacent vertices $y_1, y_2 \in Y'$ are both adjacent to x; otherwise, G_0 would induce $K_2 + K_3$ or the co-4-pan on $\{y_1, y_2, x, a, b\}$. Thus G contains all edges uv such that $u \in X$ and $v \in Y'$, and we may write $G_0 = (G_0[X \cup Y''], Y'', X) \circ G_0[\{a, b, c, d\} \cup Y']$, a contradiction, since G_0 is indecomposable.

Hence $X = \emptyset$. Since G_0 is $(K_2 + P_4)$-free, at least one of A and B must be empty, as must one of C and D. By symmetry we may assume that $B = D = \emptyset$. Since G_0 is $\{K_2 + K_3, P_5\}$-free, A and C must be independent sets, and G has no edge uv such that $u \in A$ and $v \in C$. Since G_0 is $(K_2 + P_4)$-free, no vertex of Y has a neighbor in either A or C. Thus $G_0[A \cup \{a, b\}]$ and $G_0[C \cup \{c, d\}]$ are components of G_0 that are stars. Since G_0 is $\{K_2 + K_3, K_2 + P_4, K_2 + C_4\}$-free, $G_0[Y]$ must be $\{K_3, P_4, C_4\}$-free. Note that the $\{K_3, P_4, C_4\}$-free graphs are necessarily forests with diameter at most 2 and hence are disjoint unions of stars.

Thus if G_0 induces $2K_2$, then G_0 is a disjoint union of stars. If instead G_0 induces C_4, then $\overline{G_0}$ induces $2K_2$, and by the argument above $\overline{G_0}$ is a disjoint union of stars.

We now show that (d) implies (e). As we have observed, the clique and in dependent set of a split graph form an A_4-separating partition. Since G is split if G_0 is split, we may assume that G_0 or $\overline{G_0}$ is a disjoint union of stars G'. Let A' be a maximum independent set in G', and let $B' = V(G') - A'$. Any 4-vertex induced subgraph of G' having an alternating 4-cycle is isomorphic to $2K_2$ and has a pair of nonadjacent vertices in each of A' and B'; thus A', B' is A_4-separating. If G has canonical decomposition $(G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0$, then it follows from Proposition 4.21 that the sets $A_k \cup \cdots \cup A_1 \cup A'$ and $B_k \cup \cdots \cup B_1 \cup B'$ form an A_4-separating partition of $V(G)$. The graph G is thus A_4-separable.

Finally, we show that (e) implies (a). Suppose that G is A_4-separable, and let V_1 and V_2 form an A_4-separating partition of $V(G)$. Form G' by deleting all edges of G having both endpoints in V_1 and adding every edge uv such that $u, v \in V_2$ (and uv was not already an edge in G). The graph G' is clearly a split graph, and we claim that its A_4-structure H' is the same as that of G. Indeed, each induced $2K_2$, C_4, or P_4 in G becomes an induced P_4 in G', so $E(H) \subseteq E(H')$. Conversely, consider an edge of H' arising from an alternating 4-cycle $[a, b : c, d]$ in G', where we may assume that $a \in V_1$. By Proposition 4.32 and Observation 4.33, this alternating 4-cycle occurs in an induced P_4 in G' having its endpoints in V_1 and its midpoints in V_2; thus $a, c \in V_1$ and $b, d \in V_2$. Undoing the edge additions and deletions that created G' from G cannot destroy the alternating 4-cycle $[a, b : c, d]$, so $[a, b : c, d]$ was an alternating 4-cycle in G. Thus $E(H') = E(H)$, and we have shown that G' has the same A_4-structure as the split graph G. $\square$

REFERENCES

[1] K. J. Asciak and J. Lauri. On disconnected graph with large reconstruction number. *Ars Combin.*, 62:173–181, 2002.

[2] L. Babel and S. Olariu. On the structure of graphs with few P_4s. *Discrete Appl. Math.*, 84(1-3):1–13, 1998.

[3] M. D. Barrus, M. Kumbhat, and S. G. Hartke. Graph classes characterized both by forbidden subgraphs and degree sequences. *J. Graph Theory*, 57(2):131–148, 2008.

[4] M. D. Barrus, M. Kumbhat, and S. G. Hartke. Non-minimal degree-sequence-forcing triples. Submitted, 2008.

[5] M. D. Barrus and D. B. West. The A_4-structure of a graph. Preprint, 2009.

[6] M. D. Barrus and D. B. West. On the degree-associated reconstruction number of a graph. Submitted, 2009.

[7] Z. Blázsik, M. Hujter, A. Pluhár, and Z. Tuza. Graphs with no induced C_4 and $2K_2$. *Discrete Math.*, 115(1-3):51–55, 1993.

[8] B. Bollobás. Almost every graph has reconstruction number three. *J. Graph Theory*, 14(1):1–4, 1990.

[9] J. A. Bondy. A graph reconstructor's manual. In *Surveys in combinatorics, 1991 (Guildford, 1991)*, volume 166 of *London Math. Soc. Lecture Note Ser.*, pages 221–252. Cambridge Univ. Press, Cambridge, 1991.

[10] J. A. Bondy and R. L. Hemminger. Graph reconstruction—a survey. *J. Graph Theory*, 1(3):227–268, 1977.

[11] A. Brandstädt and V. B. Le. Split-perfect graphs: characterizations and algorithmic use. *SIAM J. Discrete Math.*, 17(3):341–360 (electronic), 2004.

[12] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Ann. of Math. (2)*, 164(1):51–229, 2006.

[13] V. Chvátal. A semistrong perfect graph conjecture. In *Topics on perfect graphs*, volume 88 of *North-Holland Math. Stud.*, pages 279–280. North-Holland, Amsterdam, 1984.

[14] V. Chvátal and P. L. Hammer. Aggregation of inequalities in integer programming. In *Studies in integer programming (Proc. Workshop, Bonn, 1975)*, pages 145–162. Ann. of Discrete Math., Vol. 1. North-Holland, Amsterdam, 1977.

[15] S. Földes and P. L. Hammer. On a class of matroid-producing graphs. In *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. I*, volume 18 of *Colloq. Math. Soc. János Bolyai*, pages 331–352. North-Holland, Amsterdam, 1978.

[16] S. Földes and P. L. Hammer. On split graphs and some related questions. In *Problèmes combinatoires et théorie des graphes*, volume 260 of *Colloques Internationaux du Centre National de la Recherche Scientifique [International Colloquia of the CNRS]*, pages 139–140. Éditions du Centre National de la Recherche Scientifique (CNRS), Paris, 1978. Colloque International CNRS held at the Université d'Orsay, Orsay, July 9–13, 1976.

[17] D. R. Fulkerson, A. J. Hoffman, and M. H. McAndrew. Some properties of graphs with multiple edges. *Canad. J. Math.*, 17:166–177, 1965.

[18] T. Gallai. Transitiv orientierbare Graphen. *Acta Math. Acad. Sci. Hungar*, 18:25–66, 1967.

[19] P. L. Hammer, T. Ibaraki, and B. Simeone. Degree sequences of threshold graphs. In *Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1978)*, Congress. Numer., XXI, pages 329–355, Winnipeg, Man., 1978. Utilitas Math.

[20] P. L. Hammer and B. Simeone. The splittance of a graph. *Combinatorica*, 1(3):275–284, 1981.

[21] F. Harary and E. Palmer. The reconstruction of a tree from its maximal subtrees. *Canad. J. Math.*, 18:803–810, 1966.

[22] F. Harary and M. Plantholt. The graph reconstruction number. *J. Graph Theory*, 9(4):451–454, 1985.

[23] C. T. Hoàng. *Perfect Graphs.* PhD thesis, McGill University, Montreal, 1985.

[24] C. T. Hoàng. On the disc-structure of perfect graphs. I. The co-paw-structure. In *Proceedings of the Third International Conference on Graphs and Optimization, GO-III (Leukerbad, 1998)*, volume 94, pages 247–262, 1999.

[25] C. T. Hoàng. On the disc-structure of perfect graphs. II. The co-C_4-structure. *Discrete Math.*, 252(1-3):141–159, 2002.

[26] C. T. Hoàng and B. Reed. On the co-P_3-structure of perfect graphs. *SIAM J. Discrete Math.*, 18(3):571–576 (electronic), 2004/05.

[27] S. Hougardy. *On the P_4-structure of perfect graphs.* PhD thesis, Shaker Verlag, Aachen, 1996.

[28] S. Hougardy. The P_4-structure of perfect graphs. In *Perfect graphs*, Wiley-Intersci. Ser. Discrete Math. Optim., pages 93–112. Wiley, Chichester, 2001.

[29] B. Jamison and S. Olariu. p-components and the homogeneous decomposition of graphs. *SIAM J. Discrete Math.*, 8(3):448–463, 1995.

[30] P. J. Kelly. A congruence theorem for trees. *Pacific J. Math.*, 7:961–968, 1957.

[31] K. Kuratowski. Sur le problème des courbes gauches en topologie. *Fund. Math.*, 15:271–283, 1930.

[32] J. Lauri. Proof of Harary's conjecture on the reconstruction of trees. *Discrete Math.*, 43(1):79–90, 1983.

[33] J. Lauri. Pseudosimilarity in graphs—a survey. *Ars Combin.*, 46:77–95, 1997.

[34] L. Lovász. Normal hypergraphs and the perfect graph conjecture. *Discrete Math.*, 2(3):253–267, 1972.

[35] A. Maccari, O. Rueda, and V. Viazzi. A survey on edge reconstruction of graphs. *J. Discrete Math. Sci. Cryptography*, 5(1):1–11, 2002.

[36] F. Maffray and M. Preissmann. Linear recognition of pseudo-split graphs. *Discrete Appl. Math.*, 52(3):307–312, 1994.

[37] P. Marchioro, A. Morgana, R. Petreschi, and B. Simeone. Degree sequences of matrogenic graphs. *Discrete Math.*, 51(1):47–61, 1984.

[38] B. McMullen and S. P. Radziszowski. Graph reconstruction numbers. *J. Combin. Math. Combin. Comput.*, 62:85–96, 2007.

[39] R. Molina. Correction of a proof on the ally-reconstruction number of a disconnected graph. Correction to: "The ally-reconstruction number of a disconnected graph" [Ars Combin. **28** (1989), 123–127; MR1039138 (90m:05094)] by W. J. Myrvold. *Ars Combin.*, 40:59–64, 1995.

[40] W. Myrvold. The ally-reconstruction number of a disconnected graph. *Ars Combin.*, 28:123–127, 1989.

[41] W. Myrvold. The ally-reconstruction number of a tree with five or more vertices is three. *J. Graph Theory*, 14(2):149–166, 1990.

[42] U. N. Peled. Matroidal graphs. *Discrete Math.*, 20(3):263–286, 1977/78.

[43] N. Prince. Personal communication, 2008.

[44] S. Ramachandran. On a new digraph reconstruction conjecture. *J. Combin. Theory Ser. B*, 31(2):143–149, 1981.

[45] S. Ramachandran. Reconstruction number for Ulam's conjecture. *Ars Combin.*, 78:289–296, 2006.

[46] T. Raschle and K. Simon. On the P_4-components of graphs. *Discrete Appl. Math.*, 100(3):215–235, 2000.

[47] B. Reed. A semistrong perfect graph theorem. *J. Combin. Theory Ser. B*, 43(2):223–240, 1987.

[48] D. Seinsche. On a property of the class of n-colorable graphs. *J. Combinatorial Theory Ser. B*, 16:191–193, 1974.

[49] R. Tyshkevich. Decomposition of graphical sequences and unigraphs. *Discrete Math.*, 220(1-3):201–238, 2000.

[50] R. I. Tyshkevich. Once more on matrogenic graphs. *Discrete Math.*, 51(1):91–100, 1984.

[51] R. I. Tyškevič. Canonical decomposition of a graph. *Dokl. Akad. Nauk BSSR*, 24(8):677–679, 763, 1980.

[52] S. M. Ulam. *A collection of mathematical problems*. Interscience Tracts in Pure and Applied Mathematics, no. 8. Interscience Publishers, New York-London, 1960.

[53] D. B. West. *Introduction to graph theory*. Prentice Hall Inc., Upper Saddle River, NJ, 1996.

AUTHOR'S BIOGRAPHY

Michael David Barrus was born in Provo, Utah. He received B.S. and M.S. degrees in mathematics from Brigham Young University in 2002 and 2004. In the fall of 2004, he entered the Ph.D. program in mathematics at the University of Illinois at Urbana-Champaign, where he completed a M.S. degree in mathematics teaching in 2008 and wrote his doctoral thesis under the direction of Douglas B. West. Following the completion of his Ph.D. degree in May 2009, Barrus will begin work as an assistant professor of mathematics at Black Hills State University in Spearfish, South Dakota.