Antimagic labeling and canonical decomposition of graphs

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Abstract

An antimagic labeling of a connected graph with \( m \) edges is an injective assignment of labels from \( \{1, \ldots, m\} \) to the edges such that the sums of incident labels are distinct at distinct vertices. Hartsfield and Ringel conjectured that every connected graph other than \( K_2 \) has an antimagic labeling. We prove this for the classes of split graphs and graphs decomposable under the canonical decomposition introduced by Tyshkevich. As a consequence, we provide a sufficient condition on graph degree sequences to guarantee an antimagic labeling.

Keywords: antimagic labeling, split graph, canonical decomposition, combinatorial problems

1 Introduction

Let \( G \) be a graph with \( m \) edges. For an injective labeling of the edges of \( G \) with the labels \( 1, 2, \ldots, m \), we define \( f \) on the vertex set of \( G \) by setting \( f(v) \) to be the sum of the labels on edges containing \( v \). If \( f \) is an injective function, then we say that both the edge labeling and \( G \) are antimagic. Hartsfield and Ringel \cite{6} conjectured that every connected graph other than \( K_2 \) has an antimagic labeling. Various classes of graphs have been shown to be antimagic (see \cite{3}, \cite{4}, \cite{6}, \cite{7}, \cite{10}, and \cite{11}). In particular, Alon et al. \cite{1} showed that \( n \)-vertex graphs with maximum degree at least \( n - 2 \) and graphs with high minimum degree are antimagic.

In this paper we present an algorithm that produces an antimagic labeling for any graph containing a clique with special neighborhood properties. We characterize the graphs having such cliques; these are precisely the split graphs and graphs that are decomposable under what has been termed the canonical decomposition. As a result, we obtain a condition on the degree sequence of a graph that ensures that the graph is antimagic.

2 A dominating clique condition

We use \( V(G) \) and \( d_G(v) \) to denote the vertex set of \( G \) and the degree of vertex \( v \) in \( G \). We define the open and closed neighborhoods of a vertex \( v \) in \( G \) to be the sets

\[
N_G(v) = \{ u \in V(G) : uv \in E(G) \},
\]

\[
N_G[v] = N_G(v) \cup \{v\},
\]

respectively. Given \( W \subseteq V(G) \), let \( G[W] \) denote the induced subgraph of \( G \) with vertex set \( W \). A clique in a graph is a set of pairwise adjacent vertices; an independent set is a set of pairwise non-adjacent vertices. A split graph is a graph whose vertex set can be partitioned into a clique and an independent set.

**Lemma 1.** Let \( G \) be a connected graph on at least \( 3 \) vertices. If \( G \) has a clique \( B \) such that for every vertex \( v \) in \( G \) either \( N_G(v) \subseteq B \) or \( B \subseteq N_G[v] \), then \( G \) is antimagic.
Proof. We provide an antimagic labeling of the edges of $G$. Let $A$ denote the set of vertices $v$ not in $B$ such that $N_G(v) \subseteq B$. Let $A = \{a_1, \ldots, a_{|A|}\}$ and $B = \{b_1, \ldots, b_{|B|}\}$, where in each set the vertices are indexed in nondecreasing order of degrees, and let $C = V(G) - A - B$. Note that $A$ is an independent set, and each vertex of $C$ is adjacent to every vertex of $B$.

To begin the labeling, order the edges of the form $a_ib_j$ lexicographically on the index pairs $(i,j)$ and assign them the first numbers in order. Next, label all edges in $G[C]$ arbitrarily with the next smallest numbers. Define a function $g$ on $C$ by letting $g(v)$ denote the sum of labels on all edges incident with $v$ in $G[C]$, and denote the vertices of $C$ by $c_1, \ldots, c_{|C|}$, where the vertices are indexed in nondecreasing order of their values under $g$. Now order the edges of the form $c_ib_j$ lexicographically on the pairs $(i,j)$, and label them with the next smallest numbers. For vertices $v$ in $B$, define $g'(v)$ to be the sum of the labels on all edges joining $v$ with a vertex outside $B$ (at this point all such edges have been labeled). Denote the vertices of $B$ now as $b'_1, \ldots, b'_{|B|}$, indexing them in nondecreasing order of their values under $g'$. Label the remaining edges $b'_ib'_j$ in lexicographic order on the pairs $(i,j)$.

We prove now that the labeling described is antimagic. For each vertex $v$ in $G$, let $f(v)$ denote the sum of the labels on edges containing $v$. For $c \in C$ and $a_i, b_j \in A$ with $i < j$, note that $d_G(a_i) \leq d_G(a_j) < d_G(c)$, and if $\ell_1, \ell_2, \ell_3$ are arbitrary labels on edges incident with $a_i, a_j, c$, respectively, then $\ell_1 < \ell_2 < \ell_3$. It follows that $f(a_i) < f(a_j) < f(c)$, so $f$ is injective on $A$, and $f$ takes on different values for any two vertices $a \in A$ and $c \in C$.

For $c \in C$, note that $f(c)$ is the sum of $g(c)$ and the labels on all edges $cb$ for $b \in B$. For $c_i, c_j \in C$ with $i < j$, we have $g(c_i) \leq g(c_j)$, and for each $b \in B$ the edge $c_ib$ receives a label strictly less than the label on $c_jb$; hence $f(c_i) < f(c_j)$, so $f$ is injective on $C$. Furthermore, for $u \in A \cup C$ and $b \in B$, if $v$ is a neighbor of $u$ other than $b$, then $v$ is also a neighbor of $b$, and edge $uv$ receives a label less than the label on $bv$; hence $f(u) < f(b)$. If $u$ has no neighbors other than $b$, then $b$ has a neighbor other than $u$ (since $G$ is connected and not $K_2$) and again we see that $f(u) < f(b)$.

Finally, we show that $f$ takes on different values for all vertices of $B$. This is trivial if $|B| = 1$. Suppose that $|B| = 2$. In this case $A \cup C$ is nonempty, since $G$ is not $K_2$. If $A$ is nonempty, then it consists of pendant vertices adjacent to either $b_1$ or $b_2$. Since $d_G(b_1) \leq d_G(b_2)$, the contribution to $f(b_1)$ from edges joining $b_1$ to vertices in $A$ is strictly less than the corresponding contribution to $f(b_2)$, by construction. Each vertex in $C$ is adjacent to both $b_1$ and $b_2$, and the label on the edge joining it to $b_1$ is smaller than the label on the edge joining it to $b_2$. Since $A \cup C$ is nonempty, it follows that $f(b_1) < f(b_2)$.

Finally, suppose that $|B| \geq 3$. Let $b'_i$ and $b'_j$ be vertices of $B$ with $i < j$; by definition, $g'(b'_i) \leq g'(b'_j)$. Since every other vertex $b'_k$ in $B$ is adjacent to both $b'_i$ and $b'_j$, with edge $b'_ib'_k$ receiving a lesser label than $b'_ib'_k$, it is clear that $f(b'_i) < f(b'_j)$.

It was shown in [1] that graphs with a dominating vertex are antimagic; Lemma 1 extends this result to graphs having a special dominating clique.

Following the work of Tyshkevich in [8] (see also [9]), we define a binary operation $\circ$ with two inputs. The first input is a split graph $S$ with a given partition of its vertex set into an independent set $A$ and a clique $B$ (denote this by $S(A,B)$), and the second is an arbitrary graph $H$. The composition $S(A,B) \circ H$ is defined to be the graph resulting from taking the disjoint union of $S(A,B)$ and $H$ and adding to it all edges having an endpoint in each of $B$ and $V(H)$. For example, taking $P_4$ as the split graph (with the unique partition of its vertex set into a clique and an independent set) and $K_3$ as the second input, the composition is the graph shown in Figure 1. If $G$ contains nonempty induced subgraphs $H$ and $S$ and vertex subsets $A$ and $B$ such that $G = S(A,B) \circ H$, then $G$ is (canonically) decomposable; otherwise $G$ is indecomposable. Tyshkevich showed in [8] that each graph can be expressed as a composition $S_1(A_1, B_1) \circ \cdots \circ S_k(A_k, B_k) \circ H$ of indecomposable induced subgraphs (note that $\circ$ is associative); indecomposable graphs are those for which $k = 0$. This representation is known as the canonical decomposition of the graph and is unique up to isomorphism of the indecomposable subgraphs involved.

Lemma 2. The following are equivalent for a graph $G$:

1. $G$ has a clique $B$ such that for all $v \in V(G)$ either $N_G(v) \subseteq B$ or $B \subseteq N_G(v)$;
(2) $G$ is split or canonically decomposable.

Proof. (1) $\Rightarrow$ (2): Let $A$ be the set of vertices $v$ not in $B$ such that $N_G[v] \subseteq B$, and let $C = V(G) - A - B$. Note that $A$ is an independent set. If $C = \emptyset$, then $G$ is a split graph. Otherwise, we may write $G$ as the composition $G'(A, B) \circ G[C]$, where $G' = G[A \cup B]$.

(2) $\Rightarrow$ (1): If $G$ is split, then we may partition $V(G)$ into an independent set $A$ and a clique $B$. If $G$ is decomposable in the canonical decomposition, then we may write $G = S(A, B) \circ H$ for vertex subsets $A$ and $B$ and induced subgraphs $S$ and $H$ of $G$. In both cases either $N_G(v) \subseteq B$ or $B \subseteq N_G[v]$ for each vertex $v$ in $G$.

The previous two lemmas immediately establish our main result.

**Theorem 3.** Connected graphs on at least 3 vertices that are split or canonically decomposable are antimagic.

Observe that canonically decomposable graphs on at least three vertices are connected if and only if they have no isolated vertices. As a consequence of known degree sequence characterizations of split graphs [5, Theorem 4] and indecomposable graphs [9, Theorem 2], we have a sufficient degree sequence condition for antimagic graphs:

**Corollary 4.** Let $G$ be an $n$-vertex graph $(n \geq 3)$ with degree sequence $(d_1, \ldots, d_n)$ in nonincreasing order such that $d_n > 0$. If there exist integers $p$ and $q$ such that $0 < p + q \leq n$ and

$$\sum_{i=1}^p d_i = p(n - q - 1) + \sum_{i=n-q+1}^n d_i,$$

then $G$ is antimagic.

3 Conclusion

Theorem 3 shows that to settle the conjecture of Hartsfield and Ringel, it suffices to consider graphs that are indecomposable under the canonical decomposition. Barrus and West [2] gave a characterization of indecomposable graphs in terms of alternating 4-cycles, configurations on four vertices $a, b, c$, and $d$ such that $ab$ and $cd$ are edges and $ad$ and $bc$ are not. They showed the following.

**Theorem 5.** A graph $G$ is indecomposable under the canonical decomposition if and only if for every pair $u, v$ of vertices there is a sequence $V_1, \ldots, V_k$ of 4-element subsets of $V(G)$ such that $u$ and $v$ belong to $V_1$ and $V_k$, respectively, $V_i \cap V_{i+1} \neq \emptyset$ for $1 \leq i \leq k - 1$, and each $V_i$ is the vertex set of an alternating 4-cycle.

We conclude with the following question.

**Question.** Are there large families of connected graphs for which it is possible to use the structure of the alternating 4-cycles in the graph to produce an antimagic labeling?
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References


