Antimagic labeling and canonical decomposition of graphs

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Abstract

An antimagic labeling of a connected graph with m edges is an injective assignment of labels from $\{1,\dots,m\}$ to the edges such that the sums of incident labels are distinct at distinct vertices. Hartsfield and Ringel conjectured that every connected graph other than K_2 has an antimagic labeling. We prove this for the classes of split graphs and graphs decomposable under the canonical decomposition introduced by Tyshkevich. As a consequence, we provide a sufficient condition on graph degree sequences to guarantee an antimagic labeling.

Keywords: antimagic labeling, split graph, canonical decomposition, combinatorial problems

1 Introduction

Let G be a graph with m edges. For an injective labeling of the edges of G with the labels $1, 2, \dots, m$, we define f on the vertex set of G by setting $f(v)$ to be the sum of the labels on edges containing v. If f is an injective function, then we say that both the edge labeling and G are *antimagic*. Hartsfield and Ringel [6] conjectured that every connected graph other than K_2 has an antimagic labeling. Various classes of graphs have been shown to be antimagic (see [3], [4], [6], [7], [10], and [11]). In particular, Alon et al. [1] showed that n-vertex graphs with maximum degree at least $n-2$ and graphs with high minimum degree are antimagic.

In this paper we present an algorithm that produces an antimagic labeling for any graph containing a clique with special neighborhood properties. We characterize the graphs having such cliques; these are precisely the split graphs and graphs that are decomposable under what has been termed the *canonical decomposition*. As a result, we obtain a condition on the degree sequence of a graph that ensures that the graph is antimagic.

2 A dominating clique condition

We use $V(G)$ and $d_G(v)$ to denote the vertex set of G and the degree of vertex v in G. We define the open and closed neighborhoods of a vertex v in G to be the sets

$$N_G(v) = \{u \in V(G) \colon uv \in E(G)\},$$
$$N_G[v] = N_G(v) \cup \{v\},$$

respectively. Given $W \subseteq V(G)$, let $G[W]$ denote the induced subgraph of G with vertex set W. A *clique* in a graph is a set of pairwise adjacent vertices; an *independent set* is a set of pairwise non-adjacent vertices. A *split graph* is a graph whose vertex set can be partitioned into a clique and an independent set.

Lemma 1. *Let G be a connected graph on at least 3 vertices. If G has a clique B such that for every vertex v in G either $N_G(v) \subseteq B$ or $B \subseteq N_G[v]$, then G is antimagic.*

Proof. We provide an antimagic labeling of the edges of G. Let A denote the set of vertices v not in B such that $N_G(v) \subsetneq B$. Let $A = \{a_1, \ldots, a_{|A|}\}$ and $B = \{b_1, \ldots, b_{|B|}\}$, where in each set the vertices are indexed in nondecreasing order of degrees, and let $C = V(G) - A - B$. Note that A is an independent set, and each vertex of C is adjacent to every vertex of B.

To begin the labeling, order the edges of the form $a_i b_j$ lexicographically on the index pairs (i, j) and assign them the first numbers in order. Next, label all edges in $G[C]$ arbitrarily with the next smallest numbers. Define a function g on C by letting $g(v)$ denote the sum of labels on all edges incident with v in $G[C]$, and denote the vertices of C by $c_1, \ldots, c_{|C|}$, where the vertices are indexed in nondecreasing order of their values under g. Now order the edges of the form $c_i b_j$ lexicographically on the pairs (i, j), and label them with the next smallest numbers. For vertices v in B, define $g'(v)$ to be the sum of the labels on all edges joining v with a vertex outside B (at this point all such edges have been labeled). Denote the vertices of B now as $b'_1, \ldots, b'_{|B|}$, indexing them in nondecreasing order of their values under g'. Label the remaining edges $b'_i b'_j$ in lexicographic order on the pairs (i, j).

We prove now that the labeling described is antimagic. For each vertex v in G, let $f(v)$ denote the sum of the labels on edges containing v. For $c \in C$ and $a_i, a_j \in A$ with $i < j$, note that $d_G(a_i) \leq d_G(a_j) < d_G(c)$, and if ℓ_1, ℓ_2, ℓ_3 are arbitrary labels on edges incident with a_i, a_j, c, respectively, then $\ell_1 < \ell_2 < \ell_3$. It follows that $f(a_i) < f(a_j) < f(c)$, so f is injective on A, and f takes on different values for any two vertices $a \in A$ and $c \in C$.

For $c \in C$, note that $f(c)$ is the sum of $g(c)$ and the labels on all edges cb for $b \in B$. For $c_i, c_j \in C$ with $i < j$, we have $g(c_i) \leq g(c_j)$, and for each $b \in B$ the edge $c_i b$ receives a label strictly less than the label on $c_j b$; hence $f(c_i) < f(c_j)$, so f is injective on C. Furthermore, for $u \in A \cup C$ and $b \in B$, if v is a neighbor of u other than b, then v is also a neighbor of b, and edge uv receives a label less than the label on bv; hence $f(u) < f(b)$. If u has no neighbors other than b, then b has a neighbor other than u (since G is connected and not K_2) and again we see that $f(u) < f(b)$.

Finally, we show that f takes on different values for all vertices of B. This is trivial if $|B| = 1$. Suppose that $|B| = 2$. In this case $A \cup C$ is nonempty, since G is not K_2. If A is nonempty, then it consists of pendant vertices adjacent to either b_1 or b_2. Since $d_G(b_1) \leq d_G(b_2)$, the contribution to $f(b_1)$ from edges joining b_1 to vertices in A is strictly less than the corresponding contribution to $f(b_2)$, by construction. Each vertex in C is adjacent to both b_1 and b_2, and the label on the edge joining it to b_1 is smaller than the label on the edge joining it to b_2. Since $A \cup C$ is nonempty, it follows that $f(b_1) < f(b_2)$.

Finally, suppose that $|B| \geq 3$. Let b'_i and b'_j be vertices of B with $i < j$; by definition, $g'(b'_i) \leq g'(b'_j)$. Since every other vertex b'_k in B is adjacent to both b'_i and b'_j, with edge $b'_i b'_k$ receiving a lesser label than $b'_j b'_k$, it is clear that $f(b'_i) < f(b'_j)$. □

It was shown in [1] that graphs with a dominating vertex are antimagic; Lemma 1 extends this result to graphs having a special dominating clique.

Following the work of Tyshkevich in [8] (see also [9]), we define a binary operation $\circ$ with two inputs. The first input is a split graph S with a given partition of its vertex set into an independent set A and a clique B (denote this by $S(A, B)$), and the second is an arbitrary graph H. The *composition* $S(A, B) \circ H$ is defined to be the graph resulting from taking the disjoint union of $S(A, B)$ and H and adding to it all edges having an endpoint in each of B and $V(H)$. For example, taking P_4 as the split graph (with the unique partition of its vertex set into a clique and an independent set) and K_3 as the second input, the composition is the graph shown in Figure 1. If G contains nonempty induced subgraphs H and S and vertex subsets A and B such that $G = S(A, B) \circ H$, then G is *(canonically) decomposable*; otherwise G is *indecomposable*. Tyshkevich showed in [8] that each graph can be expressed as a composition $S_1(A_1, B_1) \circ \cdots \circ S_k(A_k, B_k) \circ H$ of indecomposable induced subgraphs (note that $\circ$ is associative); indecomposable graphs are those for which $k = 0$. This representation is known as the *canonical decomposition* of the graph and is unique up to isomorphism of the indecomposable subgraphs involved.

Lemma 2. *The following are equivalent for a graph G:*

(1) *G has a clique B such that for all $v \in V(G)$ either $N_G(v) \subseteq B$ or $B \subseteq N_G[v]$;*

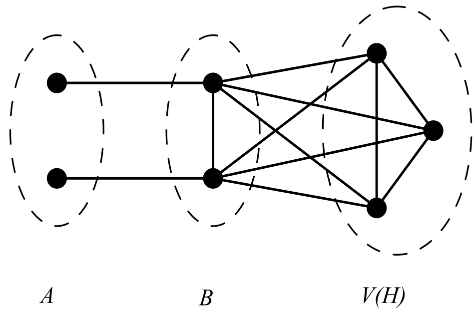

Figure 1: The composition of P_4 and K_3

(2) *G is split or canonically decomposable.*

Proof. (1) $\Rightarrow$ (2): Let A be the set of vertices v not in B such that $N_G[v] \subsetneq B$, and let $C = V(G) - A - B$. Note that A is an independent set. If $C = \emptyset$, then G is a split graph. Otherwise, we may write G as the composition $G'(A, B) \circ G[C]$, where $G' = G[A \cup B]$.
(2) $\Rightarrow$ (1): If G is split, then we may partition $V(G)$ into an independent set A and a clique B. If G is decomposable in the canonical decomposition, then we may write $G = S(A, B) \circ H$ for vertex subsets A and B and induced subgraphs S and H of G. In both cases either $N_G(v) \subseteq B$ or $B \subseteq N_G[v]$ for each vertex v in G. □

The previous two lemmas immediately establish our main result.

Theorem 3. *Connected graphs on at least 3 vertices that are split or canonically decomposable are antimagic.*

Observe that canonically decomposable graphs on at least three vertices are connected if and only if they have no isolated vertices. As a consequence of known degree sequence characterizations of split graphs [5, Theorem 4] and indecomposable graphs [9, Theorem 2], we have a sufficient degree sequence condition for antimagic graphs:

Corollary 4. *Let G be an n-vertex graph ($n \geq 3$) with degree sequence $(d_1, \dots, d_n)$ in nonincreasing order such that $d_n > 0$. If there exist integers p and q such that $0 < p + q \leq n$ and*

$$\sum_{i=1}^{p} d_i = p(n - q - 1) + \sum_{i=n-q+1}^{n} d_i,$$

then G is antimagic.

3 Conclusion

Theorem 3 shows that to settle the conjecture of Hartsfield and Ringel, it suffices to consider graphs that are indecomposable under the canonical decomposition. Barrus and West [2] gave a characterization of indecomposable graphs in terms of *alternating 4-cycles*, configurations on four vertices a, b, c, and d such that ab and cd are edges and ad and bc are not. They showed the following.

Theorem 5. *A graph G is indecomposable under the canonical decomposition if and only if for every pair u, v of vertices there is a sequence $V_1, \dots, V_k$ of 4-element subsets of $V(G)$ such that u and v belong to V_1 and V_k, respectively, $V_i \cap V_{i+1} \neq \emptyset$ for $1 \leq i \leq k - 1$, and each V_i is the vertex set of an alternating 4-cycle.*

We conclude with the following question.

Question. *Are there large families of connected graphs for which it is possible to use the structure of the alternating 4-cycles in the graph to produce an antimagic labeling?*

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