The A_4-structure of a graph

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July 27, 2010

Abstract

We define the A_4-*structure* of a graph G to be the 4-uniform hypergraph on the vertex set of G whose edges are the vertex subsets inducing $2K_2$, C_4, or P_4. We show that perfection of a graph is determined by its A_4-structure. Also, the A_4-structure is naturally related to the canonical decomposition of a graph as defined by Tyshkevich [Discrete Mathematics 220 (2000) 201-238]; in particular, a graph is indecomposable if and only if its A_4-structure is connected. We also characterize those graphs having the same A_4-structure as a split graph.

1 Introduction

The P_4-*structure* of a simple graph G is the 4-uniform hypergraph with the same vertex set as G whose edges are the vertex subsets inducing a 4-vertex path. Chvátal [4] defined the P_4-structure in 1984 in studying the complexity of recognizing perfect graphs. Since its introduction, the P_4-structure has also been used in refinements of the modular decomposition of a graph (see [16] and [20]) and in defining or characterizing several classes of graphs (see [2] for a hierarchy of graph classes defined in terms of their P_4-structure).

If $\mathcal{F}$ is any set of unlabeled graphs, we may similarly define the $\mathcal{F}$-structure of a graph G as the hypergraph on the vertex set of G having as edges the vertex subsets that induce elements of $\mathcal{F}$. Such structures have been studied when $\mathcal{F}$ is $\{P_3\}$, $\{C_5, \overline{P_3 + K_1}, P_3 + K_1\}$, $\{2K_2, C_4, C_5\}$, $\{P_3, K_2 + K_1\}$, and $\{K_3, 3K_1\}$ (see [11], [12], [13], [14], and [15]). (The disjoint union and the join of graphs G and H will be denoted by $G + H$ and $G \vee H$, respectively.)

In this paper we consider the A_4-*structure* of a graph G, which we define as the 4-uniform hypergraph on the vertex set of G having as edges those vertex subsets that induce an element of $\{2K_2, C_4, P_4\}$. This terminology reflects that $2K_2, C_4$, and P_4 are the 4-vertex graphs that contain an *alternating 4-cycle*, which is a configuration on four vertices a, b, c, d in which ab and cd are edges while bc and ad are not. An alternating 4-cycle and the three resulting graphs appear in Figure 1, where dashed segments denote required non-adjacencies.
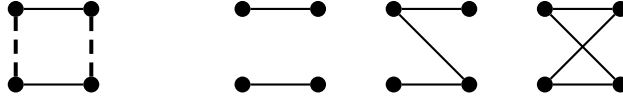
Figure 1: Alternating 4-cycle and A_4-graphs

We have several motivations for defining a hypergraph in terms of alternating 4-cycles. First, alternating 4-cycles are a fundamental notion in the study of degree sequences. A *2-switch* is a graph transformation that exchanges the edges and non-edges of an alternating 4-cycle. This usually changes the isomorphism class, but it does not change the degree sequence. In fact, a well known result of Fulkerson, Hoffman, and McAndrew [8] states that two unlabeled graphs have the same degree sequence if and only if one can be changed into the other via 2-switches. Thus we may expect to find relationships between the A_4-structure and the degree sequence of a graph.

A second motivation is the relation of alternating 4-cycles to the *canonical decomposition* of a graph, defined by Tyshkevich in [23] (see also [24]). We explore this relation in Section 3, showing that a graph is indecomposable if and only if its A_4-structure is a connected hypergraph. In Section 4, we show that the A_4-structure is to the canonical decomposition as the P_4-structure of a graph is to a refinement of its modular decomposition.

Our third motivation is the role of alternating 4-cycles in characterizing threshold graphs, matroidal graphs, and matrogenic graphs. Threshold graphs were introduced in [5] in connection with set-packing problems and have been rediscovered several times (see [18] for the history). One of the many known characterizations is in terms of the A_4-structure. Threshold graphs are precisely those graphs having no alternating 4-cycle [5], which is the statement that the A_4-structure has no edges.

A graph G is *matroidal* if the edge pairs appearing in alternating 4-cycles are the circuits of a matroid on $E(G)$. These were introduced and characterized in [19] as the graphs not containing an induced 5-cycle or the configuration $\mathcal{C}$ in Figure 2, where dashed segments join vertices required to be nonadjacent; $\mathcal{C}$ consists of a vertex triple S and vertices $x, y \notin S$ such that $N(x) \cap S$ and $N(y) \cap S$ partition S into two nonempty sets.
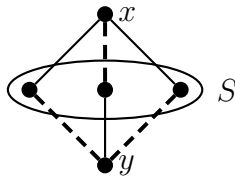

Figure 2: The configuration $\mathcal{C}$

A graph G is *matrogenic* if the vertex sets of alternating 4-cycles are the circuits of a matroid on $V(G)$. These were introduced and characterized in [7] as the graphs forbidding $\mathcal{C}$ (but allowing induced 5-cycles). Among graphs on five vertices, C_5 is the only one having more than three edges in its A_4-structure, and the graphs in which $\mathcal{C}$ appears are those whose A_4-structures have two or three edges.

Using $\mathcal{C}$, both classes are characterized by their A_4-structures.

Observation 1.1. *A graph is matrogenic if and only if no five of its vertices induce exactly two or three edges in the A_4-structure. A graph is matroidal if and only if no five of its vertices induce more than one edge in the A_4-structure.*

The (q, t)-*graphs* were defined in [1] as graphs in which no q vertices induce more than t copies of P_4; the P_4-free graphs are the $(4, 0)$-graphs, and the $(5, 1)$-graphs have been studied as P_4-*sparse* graphs [10]. Letting $[q, t]$-*graphs* be those in which no q vertices induce more than t edges in the A_4-structure, the threshold graphs become the $[4, 0]$-graphs, and by Observation 1.1 the matroidal graphs are the $[5, 1]$-graphs.

In Section 2, we show that the only graphs having the same A_4-structure as a cycle are the cycle and its complement (except for cycles with length in $\{3, 4, 6\}$); thus membership in the class of perfect graphs is determined by the A_4-structure. In a triangle-free graph, it also determines which induced subgraphs (with more than two vertices) have perfect matchings. In Section 5 the problem of obtaining all realizations of a given A_4-structure motivates our characterization of A_4-*split graphs*, which are the graphs G such that some split graph has the same A_4-structure as G. A *split graph* is a graph whose vertex set can be partitioned into a clique (a set of pairwise adjacent vertices) and an independent set (a set of pairwise nonadjacent vertices).

We use $V(G)$ and $E(G)$ to denote the vertex and edge sets of a graph G (no loops or multiple edges). Let $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$. Let $d_G(v) = |N_G(v)|$. A vertex v in G is *isolated* if $d_G(v) = 0$, *pendant* if $d_G(v) = 1$, and *dominating* if $N_G[v] = V(G)$. Given a subset S of $V(G)$, the subgraph $G[S]$ *induced by* S is the graph with vertex set S in which two vertices are adjacent if and only if they are adjacent in G. Let $G - v = G[V(G) - v]$. Given a set $\mathcal{F}$ of graphs, G is $\mathcal{F}$-*free* if no induced subgraph of G is isomorphic to an element of $\mathcal{F}$.

2 A_4-structure and cycles

Given an A_4-structure H, any graph whose A_4-structure is (isomorphic to) H is a *realization* of H. Given graphs G and G' with A_4-structures H and H', an A_4-*isomorphism* from G to G' is a bijection $\varphi \colon V(G) \to V(G')$ that is an isomorphism from H to H'. If an A_4-isomorphism exists from G to G', then we say that G and G' have *the same A_4-structure* or are A_4-*isomorphic*. Our main result in this section characterizes the graphs having the same A_4-structure as a cycle. We begin with several observations.

Observation 2.1. *If four vertices induce an alternating 4-cycle in a graph, then they also induce an alternating 4-cycle in the complement of the graph. Hence a graph and its complement have the same (labeled) A_4-structure.*

Observation 2.2. *Four vertices form an edge in the A_4-structure of a graph G (a) if and only if the subgraph they induce in G has no vertex of degree 0 or 3. (b) if and only if the induced subgraph has no triangle or independent 3-set.*

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In [4], Chvátal showed that odd cycles of length at least 5 and their complements are the only realizations of their respective P_4-structures, and he conjectured that two graphs with the same P_4-structure are either both perfect or both imperfect. Reed [21] proved this. The statement is now known as the Semistrong Perfect Graph Theorem, since it implies the Perfect Graph Theorem of Lovász [17] and is in turn implied by the Strong Perfect Graph Conjecture, proved much later by Chudnovsky et al. [3].

Strong Perfect Graph Theorem ([3]). *A graph G is perfect if and only if it induces no odd cycle of length at least 5 or the complement of such a cycle.*

Motivated by the results of Chvátal and Reed, we show that the cycle C_n and its complement are the only realizations of their respective A_4-structures if and only if $n \notin \{3, 4, 6\}$. By the Strong Perfect Graph Theorem, it then follows that graphs with the same A_4-structures are either both perfect or both imperfect.

The claim follows by inspection for $n \leq 6$, so we restrict to $n \geq 7$ and consider the cycle with vertices $u_1, \ldots, u_n$ in order. Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$ and A_4-structure H such that mapping u_i to v_i for all i defines an A_4-isomorphism. Furthermore, since both G and $\overline{G}$ are A_4-isomorphic to C_n under this map, we may assume that $v_1 v_2 \in E(G)$. We will prove that the map is a graph isomorphism. In comparing the A_4-structures of C_n and G, all addition and subtraction in indices of vertices will be done modulo n.

Lemma 2.3. *No triangle or independent 3-set in G contains two vertices with (cyclically) consecutive indices.*

Proof. From the A_4-structure of C_n, for any $i, j \in \{1, \ldots, n\}$ there is some edge of the A_4-structure of G containing v_i, v_{i+1}, and v_j. By Observation 2.2(b), these vertices induce no triangle or $3K_1$ in G. $\square$

Note that the A_4-structure of C_n (and G) is the set of all 4-tuples consisting of two pairs of consecutively-indexed vertices.

Lemma 2.4. *There is no index i such that v_i is adjacent in G to exactly one of $\{v_{i-1}, v_{i+1}\}$.*

Proof. We argue by contradiction. By symmetry in the indices, we may assume that $v_{i-1} v_i \in E(G)$ and $v_i v_{i+1} \notin E(G)$. By symmetry in G and $\overline{G}$, we may assume that $v_{i+1} v_{i-1} \notin E(G)$. Since $\{v_{i-2}, v_{i-1}, v_i, v_{i+1}\}$ and $\{v_{i-1}, v_i, v_{i+1}, v_{i+2}\}$ are both edges in the A_4-structure H, Observation 2.2(a) requires $v_{i-2}, v_{i+2} \in N_G(v_{i+1})$. Since $\{v_{i-3}, v_{i-2}, v_{i+1}, v_{i+2}\}$ and $\{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\}$ are also edges of H, Observation 2.2(a) and (b) respectively forbid $v_{i+1} v_{i-3}$ and $v_{i-2} v_{i+2}$ from $E(G)$. The present status is shown in Figure 3.

Since $\{v_{i-3}, v_{i-2}, v_i, v_{i+1}\}$ is an edge in H but v_{i+1} is not adjacent to v_{i-3} or v_i, Observation 2.2(b) requires $v_{i-3} v_i \in E(G)$. Since $n \geq 7$, $\{v_{i-2}, v_i, v_{i+1}, v_{i+2}\}$ is not an edge in H, and hence $v_i v_{i+2} \notin E(G)$. Now $\{v_i, v_{i-3}, v_{i+1}, v_{i+2}\}$ is an edge in H, a contradiction. $\square$

For an alternating 4-cycle, we use the notation $[a, b : c, d]$, indicating that $ab, cd \in E(G)$ and $bc, da \notin E(G)$.
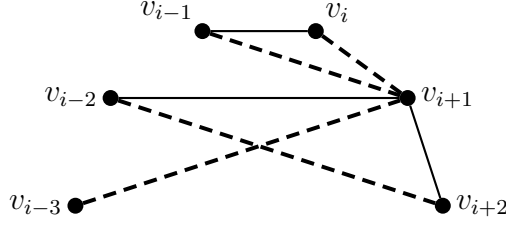
Figure 3: The subgraph of G from Lemma 2.4.

Theorem 2.5. *For $n \geq 7$, if G has the same A_4-structure as C_n, then G is isomorphic to C_n or $\overline{C}_n$.*

Proof. Since G and its complement have the same A_4-structure, we may assume that $v_1 v_2 \in E(G)$. We complete the proof by showing that G has no edges of the form $v_i v_j$ where i and j are not cyclically consecutive. This yields $G \cong C_n$.

By Lemma 2.4, G has a spanning cycle with vertices $v_1, \ldots, v_n$ in order. By Lemma 2.3, $v_i v_{i+2} \notin E(G)$ for all i. Suppose that G has a chord $v_j v_k$ for vertices v_j and v_k at a distance of at least 3 on the cycle. By Lemma 2.3, $v_j v_{k-1}, v_j v_{k+1} \notin E(G)$. It follows that $[v_j, v_k : v_{k-2}, v_{k-1}]$ and $[v_j, v_k : v_{k+2}, v_{k+1}]$ are alternating 4-cycles in G. Since $n \geq 7$, either v_{k-2} or v_{k+2} is not consecutive to v_j, which contradicts the description of the A_4-structure of C_n. Thus G has no chords and hence is isomorphic to C_n. $\square$

Corollary 2.6. *If two graphs have the same A_4-structure, then they are either both perfect or both imperfect.*

Proof. Suppose that G and G' have the same A_4-structure, and let $\varphi \colon V(G) \to V(G')$ be an A_4-isomorphism. Let n be an odd integer such that $n \geq 5$. By Theorem 2.5, G induces C_n or $\overline{C}_n$ on a vertex subset S if and only if G' induces C_n or $\overline{C}_n$ on $\varphi(S)$. The Strong Perfect Graph Theorem then implies the result. $\square$

The conclusion of Theorem 2.5 does not hold when $n = 6$; the graph C_6 shares its A_4-structure with G' and $\overline{G'}$, where G' is any graph obtained by deleting up to three pairwise non-incident edges from $K_{3,3}$. Note also that Theorem 2.5 applies to long cycles of both parities, whereas Chvátal's analogous result for P_4-structure deals only with odd cycles.

We conclude our discussion of cycles and A_4-structure by presenting a result on matchings in triangle-free graphs. We need a lemma.

Lemma 2.7. *If G is a 6-vertex triangle-free graph whose vertex set can be partitioned into three pairs of vertices such that the union of any two of these pairs is an edge in the A_4-structure of G, then G has a perfect matching.*

Proof. Let H be the A_4-structure of G, and let A, B, and C denote the vertex pairs described, so that $V(G) = A \cup B \cup C$ and $A \cup B, A \cup C, B \cup C \in E(H)$. If the vertices in each of A, B, and C induce an edge in G, then G has a perfect matching. If not, then we may assume without loss of generality that $a_1 a_2 \notin E(G)$, where $A = \{a_1, a_2\}$. Since $A \cup B \in E(H)$, vertices a_1 and a_2 belong to non-incident edges $a_1 b_1$ and $a_2 b_2$ in $G[A \cup B]$. Similarly, there

exist non-incident edges $a_1 c_1$ and $a_2 c_2$ in $G[A \cup C]$. Since G is triangle-free, $b_1 c_1, b_2 c_2 \notin E(G)$. However, $B \cup C \in E(H)$, so $B \cup C$ induces two non-incident edges. Whether these edges are $\{b_1 b_2, c_1 c_2\}$ or $\{b_1 c_2, b_2 c_1\}$, it follows that G has a spanning cycle and hence a perfect matching. $\square$

For graphs G and G', we say that a bijection $\varphi : V(G) \to V(G')$ *preserves matchings* if a set S is the vertex set of a matching of size at least 2 in G if and only if $\varphi(S)$ is the vertex set of a matching in G'.

Theorem 2.8. *Let G and G' be triangle-free graphs, and let $\varphi : V(G) \to V(G')$ be a bijection. The map φ is an A_4-isomorphism if and only if it preserves matchings.*

Proof. Suppose that φ preserves matchings. In a triangle-free graph G, the four vertices spanned by a matching of size 2 contain no 3-clique or independent set of size 3, so by Observation 2.2(b) these vertices form an edge in the A_4-structure of G. Conversely, the three 4-vertex graphs $2K_2$, P_4, and C_4 that have an alternating 4-cycle all have perfect matchings. Thus a vertex subset S in G induces an alternating 4-cycle if and only if $\varphi(S)$ induces an alternating 4-cycle in G'; hence φ is an A_4-isomorphism.

Suppose instead that $\varphi : V(G) \to V(G')$ is an A_4-isomorphism, and let S be the vertex set of some matching in G of size at least 2. We may partition the edges of the matching on S into pairs and triples of edges; let $S_1, S_2, \ldots, S_j$ be the vertex sets of these edge sets. By the previous paragraph and Lemma 2.7, the sets $\varphi(S_i)$ are the vertex sets of disjoint matchings in G'. The union of these matchings is a matching on $\varphi(S)$, so φ preserves matchings. $\square$

3   Canonical decomposition and A_4-structure

In this section we describe the relationship between the A_4-structure of a graph and its canonical decomposition, as defined by Tyshkevich [23, 24].

A *splitted graph* is a triple (G, A, B) such that G is a split graph whose vertices partition into an independent set A and a clique B. Two splitted graphs (G, A, B) and (G', A', B') are *isomorphic* if there exists a graph isomorphism $\varphi : V(G) \to V(G')$ such that $\varphi(A) = A'$. Given a splitted graph (G, A, B) and a graph H on disjoint vertex sets, we define the *composition* of (G, A, B) and H to be the graph $(G, A, B) \circ H$ formed by adding to $G + H$ all edges uv such that $u \in B$ and $v \in V(H)$. For example, when $H = K_3$ and $G = P_4$, with A the set of endpoints and B the set of midpoints of G, the composition $(G, A, B) \circ K_3$ is the graph on the left in Figure 4 (here and henceforth, heavy lines joining sets of vertices mean that all possible edges joining the two sets are present). On the right we show $(G, A, B) \circ ((G, A, B) \circ K_3)$.

The operation $\circ$ is associative, so henceforth we omit parentheses for grouping when performing multiple compositions. Note that in a composition $(G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0$, each vertex in B_i is adjacent to every vertex in $\bigcup_{j<i} V(G_j)$, each vertex in A_i is adjacent to none of the vertices in $\bigcup_{j<i} V(G_j)$, and only the rightmost graph in the composition can fail to be a split graph. A graph is *decomposable* if it can be written as a composition

Figure 4: The compositions $(G, A, B) \circ H$ and $(G, A, B) \circ (G, A, B) \circ H$.

$(G, A, B) \circ H$, where G and H both have at least one vertex. Otherwise, it is *indecomposable*. Tyshkevich proved the following:

Theorem 3.1 (Tyshkevich [24])**.** *Every graph G can be expressed as a composition*

$$G = (G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0 \tag{$*$}$$

of indecomposable components, where the (G_i, A_i, B_i) are indecomposable splitted graphs and G_0 is a non-null indecomposable graph (if G is indecomposable, then $k = 0$). Furthermore, graphs G and G' expressed as $(*)$ and $G' = (G'_\ell, A'_\ell, B'_\ell) \circ \cdots \circ (G'_1, A'_1, B'_1) \circ G'_0$ are isomorphic if and only if $G_0 \cong G'_0$, $k = \ell$, and $(G_i, A_i, B_i) \cong (G'_i, A'_i, B'_i)$ for all i.

Theorem 3.1 states both existence and uniqueness of the decomposition into indecomposable components, so we call it the *canonical decomposition*. We next characterize indecomposable graphs in terms of their A_4-structures. A hypergraph H is *connected* if, for all $x, y \in V(H)$, there exist edges $A_0, \ldots, A_k$ such that $x \in A_0$, $y \in A_k$, and consecutive edges in the list intersect. The *distance* between vertices x and y in a connected hypergraph is the least k for which such a list exists. A *component* of H is a maximal connected hypergraph contained in H.

Theorem 3.2. *A graph is indecomposable with respect to canonical decomposition if and only if its A_4-structure is connected. Hence, the vertex sets of the G_i in the canonical decomposition $(*)$ are exactly the vertex sets of the components in the A_4-structure of G.*

The proof is lengthy, so we first prove several preliminary results.

Observation 3.3. *If a graph G has more than one vertex and has canonical decomposition $(*)$, then G has an isolated vertex or dominating vertex if and only if $k \geq 1$ and G_k has exactly one vertex. The vertex is dominating in G if $A_k = \varnothing$ and is isolated in G if $B_k = \varnothing$.*

Observation 3.4. *If $G = (G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0$, then $\overline{G} = (\overline{G_k}, B_k, A_k) \circ \cdots \circ (\overline{G_1}, B_1, A_1) \circ \overline{G_0}$.*

Proposition 3.5. *If G is an indecomposable graph with more than one vertex, then every vertex of G belongs to an alternating 4-cycle in G.*

Proof. We prove the contrapositive. Suppose that some vertex v in G belongs to no alternating 4-cycle. If v is a dominating or isolated vertex, then G is decomposable by Observation 3.3, so we may assume that v is neither. Let $V_1 = N(v)$ and $V_2 = V(G) - N[v]$.

If V_1 is not a clique, then there exist $u, w \in V_1$ such that $uw \notin E(G)$. For $a \in V_2$, since neither $[v, w : u, a]$ nor $[v, u : w, a]$ is an alternating 4-cycle (both contain v), a is adjacent to neither u nor w. Hence the vertices of V_1 having a neighbor in V_2 are pairwise adjacent; indeed, each such vertex dominates V_1. Let $B = \{x \in V_1 : N_G(x) \cap V_2 \neq \varnothing\}$.

Note also that V_2 is an independent set; if a and b were adjacent vertices in V_2, then $[v, u : a, b]$ would be an alternating 4-cycle containing v. Letting $S = V_1 - B$, we can express G as the composition $(G', V_2, B) \circ (K_1, \varnothing, \{v\}) \circ G[S]$, where $G' = G[V_2 \cup B]$. Since G has more than one vertex, at least one of V_2, B, S is nonempty, so G is decomposable. Hence we may assume that V_1 is a clique in G.

We note that the complement of an alternating 4-cycle is an alternating 4-cycle, so v also belongs to no alternating 4-cycle in $\overline{G}$. Since $N_{\overline{G}}(v) = V_2$ and $V(\overline{G}) - N_{\overline{G}}[v] = V_1$, we may similarly assume that V_2 is a clique in $\overline{G}$ and hence an independent set in G.

Since V_1 is a clique and V_2 is an independent set, with $V_1 \subseteq N(v)$, we have $G = (G', V_2, V_1) \circ G[\{v\}]$, where $G' = G[V_2 \cup V_1]$. Hence in all cases G is decomposable. $\square$

When A and B are edges in the A_4-structure of G, we write $A \to B$ to mean that $G[A] \cong P_4$, the midpoints of $G[A]$ dominate B, and the endpoints of $G[A]$ are nonadjacent to each vertex in B. When discussing relations among the edges of the A_4-structure, it is convenient to use $H(G)$ to denote the A_4-structure of a graph G.

Lemma 3.6. *If A and B are disjoint edges in $H(G)$ such that no edge of $H(G)$ intersects both A and B, then $A \to B$ or $B \to A$.*

Proof. Let $[a, b : c, d]$ and $[e, f : g, h]$ be alternating 4-cycles in $G[A]$ and $G[B]$, respectively. Since $\{a, b, e, f\} \notin E(H(G))$ and each of these vertices already has a neighbor among the other three, Observation 2.2(a) implies that one vertex in $\{a, b, e, f\}$ dominates the other three; let a be such a vertex. Since neither $[a, f : g, h]$ nor $[a, e : h, g]$ is an alternating 4-cycle in G, we have $ag, ah \in E(G)$. Thus a dominates B. It follows that d has no neighbor v in B, for otherwise $[a, u : v, d]$ would be an alternating 4-cycle, where u is the non-neighbor of v in B. Making the same argument starting with $\{c, d, g, h\}$ now implies that c dominates B (since d does not dominate B) and that b has no neighbor in B.

Finally, note that $bd \notin E(G)$ and $ac \in E(G)$, since otherwise $[b, d : e, f]$ or $[a, e : h, c]$ would be an alternating 4-cycle, respectively. We conclude that $G[A] \cong P_4$, with midpoints a and c dominating B, and endpoints b and d adjacent to no vertex of B. Thus $A \to B$.

The same conclusion holds by a symmetric argument if b dominates $\{a, e, f\}$. If instead e or f dominates the other three vertices of $\{a, b, e, f\}$, then we arrive similarly at $B \to A$. $\square$

Lemma 3.6 implies that if two vertices of G each belong to an induced $2K_2$ or C_4, then they have distance at most 3 in $H(G)$, since some edge of $H(G)$ must intersect these edges containing them. We also have the following result.

Corollary 3.7. *Let G be a graph. If A and B are edges in distinct components of $H(G)$, then $A \to B$ or $B \to A$.*

Lemma 3.8. *Let G be a graph. If A, B, and C are edges in $H(G)$ such that $A \to B$ and $A \cap C$ is nonempty, then $B \not\to C$.*

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Proof. If $B \to C$, then the midpoints of the path induced by B dominate C, and the endpoints have no neighbors in C. Hence no vertex in C can dominate or be independent of B. Since every vertex of A dominates or is independent of B, we conclude that $A \cap C = \varnothing$. $\square$

Proposition 3.9. *Let G be a graph. Let Q_1 and Q_2 be distinct components of $H(G)$, and let A and B be edges in Q_1 and Q_2, respectively. If $A \to B$, then $C \to D$ for any edges C in Q_1 and D in Q_2.*

Proof. Since $A, C \in E(Q_1)$, there are edges $R_0, \ldots, R_k \in E(Q_1)$ such that $A = R_0$, $C = R_k$, and $R_{i-1} \cap R_i \neq \varnothing$ for $1 \le i \le k$. By Corollary 3.7, $B \to R_i$ or $R_i \to B$ for each i. Inductively, from $R_i \to B$, Lemma 3.8 forbids $B \to R_{i+1}$, and hence $R_{i+1} \to B$. In particular, $C \to B$.

Similarly, since $B, D \in E(Q_2)$, there are edges $S_0, \ldots, S_\ell \in E(Q_2)$ such that $B = S_0$, $D = S_\ell$, and $S_{i-1} \cap S_i \neq \varnothing$ for $1 \le i \le \ell$. Corollary 3.7 implies that $S_i \to C$ or $C \to S_i$ for each i. If $S_1 \to C$, then Lemma 3.8 yields $C \nrightarrow B$, which is false; hence $C \to S_1$. Repeating this argument iteratively yields $C \to S_i$ for each i. Thus $C \to D$. $\square$

A *tournament* is an orientation of a complete graph.

Lemma 3.10. *Let T be a directed graph whose vertices are the components of $H(G)$, defined by putting $Q_1 Q_2 \in E(T)$ if Q_1 has an edge A and Q_2 has an edge B such that $A \to B$. If G is indecomposable, then T is a transitive tournament.*

Proof. We write $Q_1 \to Q_2$ if $Q_1 Q_2 \in E(T)$. Since G is indecomposable, Proposition 3.5 implies that each component of $H(G)$ has an edge. By Corollary 3.7, any two vertices of T are adjacent. By Proposition 3.9, $Q_1 \to Q_2$ implies $Q_2 \nrightarrow Q_1$. Hence T is a tournament.

If $Q_1 \to Q_2 \to Q_3$, then by the definition of "$\to$" there exist $A_i \in E(Q_i)$ for $i \in \{1, 2, 3\}$ such that $A_1 \to A_2 \to A_3$. We claim $A_1 \to A_3$, which again by definition yields $Q_1 \to Q_3$.

Since $A_1 \to A_2 \to A_3$, we have $G[A_1] \cong G[A_2] \cong P_4$. For $i \in \{1, 2\}$, let a_i be a leaf in $G[A_i]$, and let b_i be its neighbor in $G[A_i]$. Since Q_1, Q_2, Q_3 are distinct components, when $c \in A_3$ neither $[b_1, a_1 : b_2, c]$ nor $[b_2, a_2 : c, a_1]$ is an alternating 4-cycle. Hence $cb_1 \in E(G)$ and $ca_1 \notin E(G)$. Since a_1 may be either leaf in $G[A_1]$, we conclude that $A_1 \to A_3$. $\square$

Proposition 3.11. *For any graph G, no edge of $H(G)$ contains vertices from distinct components of the canonical decomposition of G.*

Proof. Let $(G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0$ be the canonical decomposition of G. Suppose that some alternating 4-cycle $[u, v : w, x]$ has vertices in more than one G_i. Let j be the largest index such that G_j contains some vertex of the alternating 4-cycle. If such a vertex is in B_j, then its nonneighor on the alternating 4-cycle must lie in A_j, since B_j dominates $V(G_i)$ for $i < j$. If such a vertex is in A_j, then its neighor on the alternating 4-cycle must lie in B_j, since vertices of A_j are independent of $V(G_i)$ for $i < j$. Hence following along the alternating 4-cycle implies that its vertices all lie in $V(G_j)$. $\square$

We are now ready to prove our main result:

Proof of Theorem 3.2. Given a graph G, suppose first that $H(G)$ is connected. For $u, v \in V(G)$, there exist edges $E_0, \ldots, E_k$ of $H(G)$ such that $u \in E_0$ and $v \in E_k$, and

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$E_i \cap E_{i-1} \neq \varnothing$ for $1 \leq i \leq k$. Applying Proposition 3.11 to vertices in the sets $E_0, \ldots, E_k$ in turn yields that u and v belong to the same component in the canonical decomposition of G. Thus G is indecomposable.

If G is indecomposable, then the digraph T of Lemma 3.10 is a transitive tournament. If $H(G)$ is disconnected, then T is nontrivial; let Q be the source vertex. By Proposition 3.5, every component of $H(G)$ contains an edge. Every edge of Q in $H(G)$ corresponds to an induced P_4 in G whose midpoints dominate every vertex outside $V(Q)$ and whose endpoints have neighbors (in G) only in $V(Q)$.

Therefore, no vertex in $V(Q)$ is both a midpoint of some induced P_4 and an endpoint of another, and $V(Q)$ splits into sets A and B consisting of the endpoints and the midpoints of the induced copies of P_4 in $G[V(Q)]$, respectively. Two adjacent vertices of A would create an alternating 4-cycle with any two adjacent vertices of $G - V(Q)$; similarly, two nonadjacent vertices of B would create an alternating 4-cycle with any two nonadjacent vertices of $G - V(Q)$. We conclude that B is a clique and A is an independent set in G. Hence $G = (G', A, B) \circ G[V(G) - V(Q)]$, where $G' = G[A \cup B]$, and G is decomposable.

Having shown that G is indecomposable if and only if $H(G)$ is connected, it follows that the components of H partition the set $V(G)$ into exactly the same subsets that the components in the canonical decomposition do. $\square$

Theorem 3.2 provides a connection between the A_4-structure of a graph and its degree sequence. This is not surprising, since alternating 4-cycles can be used to convert one realization of a degree sequence into another. Tyshkevich [23, 24] gave a characterization of indecomposable graphs in terms of their degree sequences that has a quick explanation in terms of these ideas.

Proposition 3.12. *Applying a 2-switch to a graph G does not change the partition of $V(G)$ given by the components of the canonical decomposition of G.*

Proof. Proposition 3.11 implies that every alternating 4-cycle C is contained within a single component G_i of the canonical decomposition. Note that C must have two vertices in the clique B_i and two in the independent set A_i. Since B_i is a clique and A_i is independent, C must alternate between B_i and A_i. Hence performing a 2-switch along C leaves B_i and A_i as a clique and an independent set, thereby preserving the components of the canonical decomposition other than G_i. Furthermore, G_i cannot change into a decomposable graph, because the vertices of C would then form an alternating 4-cycle intersecting more than one component of the decomposition. $\square$

Corollary 3.13 (Tyshkevich [23, 24]). *For every graph G, the degree sequence of G uniquely determines the number of indecomposable components in the canonical decomposition of G and the degree sequences of those components.*

Corollary 3.14. *If G and G' are graphs with the same degree sequence, then $H(G)$ and $H(G')$ have the same number of components and the same sizes of vertex sets of corresponding components.*

4 A_4-structure and modules

Expanding on the previous section, we show that A_4-structure is to canonical decomposition as P_4-structure is to other graph decompositions. In particular, we develop an analogue of the "primeval decomposition" of Jamison and Olariu [16] (see Theorem 4.8), which itself refines the well known "modular decomposition" of Gallai [9] (see Theorem 4.7).

The key is an analogue of the notion of "module". A *module* in a graph G is a nonempty set $S \subseteq V(G)$ such that every vertex outside S is adjacent to all of S or to none of S. A module S is *trivial* if $|S| = 1$ or $S = V(G)$.

Lemma 4.1 (Seinsche [22]). *The following hold for every graph G.*

  (i) *The vertex set of an induced P_4 in G and a module in G can intersect only in zero, one, or four vertices.*

 (ii) *G is P_4-free if and only if every induced subgraph with at least three vertices has a nontrivial module.*

We introduce a restricted notion of module that plays for A_4-structure the role that modules play for P_4-structure. An *alternating path* is a list of vertices whose consecutive pairs alternate being adjacent and nonadjacent; the first may equal the last, but otherwise the vertices are distinct. We write $\langle v_0, \ldots, v_p \rangle$ for an alternating path of *length p* with *endpoints* v_0 and v_p. An alternating path is *S-terminal* if it has length at least 2 and its only vertices in S are its endpoints. Modules are characterized by forbidding S-terminal alternating paths of length 2. A *strict module* is a set S in $V(G)$ such that G has no S-terminal alternating path of any length. In fact, it suffices to forbid only the short alternating paths.

Proposition 4.2. *A vertex subset S of a graph G is a strict module if and only if G has no S-terminal alternating paths of length 2 or 3.*

Proof. If S is a strict module, then by definition G has no short S-terminal alternating paths. If S is not a strict module, then G has an S-terminal alternating path; let $\langle v_0, \ldots, v_p \rangle$ be a shortest one. If $p \geq 4$, then consider v_2. Whether v_2 is adjacent to v_0 or not, we can start from v_0 and continue from v_2 to v_1 or v_3. That is, $\langle v_0, v_2, v_1, v_0 \rangle$ or $\langle v_0, v_2, v_3, \ldots, v_p \rangle$ is a shorter S-terminal alternating path. Thus $p \leq 3$. $\qquad\square$

As with modules, the full vertex set is a *trivial* strict module. Always single vertices are modules, but they need not form strict modules. Proposition 4.4 below is analogous to Lemma 4.1. Recall that the threshold graphs are the graphs having no alternating 4-cycles.

Theorem 4.3 (Chvátal–Hammer [5]). *A graph G is a threshold graph if and only if G arises from a single vertex by iteratively adding an isolated vertex or a dominating vertex.*

Proposition 4.4. *The following hold for every graph G.*

  (i) *Every alternating 4-cycle and strict module intersect in zero or four vertices.*

 (ii) *G has no alternating 4-cycles if and only if every induced subgraph with at least two vertices has a nontrivial strict module.*

Proof. (i) If a vertex set S contains one, two, or three vertices of an alternating 4-cycle C, there is an S-terminal alternating path along C, so S is not a strict module.

(ii) If G has an alternating cycle, then by (i) it has a 4-vertex induced subgraph with no nontrivial strict module. If G has no alternating cycle, then every induced subgraph G' is a threshold graph. By Theorem 4.3, G' has a dominating or isolated vertex u. Now $V(G') - \{u\}$ is a strict module in G'. □

A strict module in G yields a composition for G in the sense of Tyshkevich.

Proposition 4.5. *Let S be a strict module in a graph G. If A and B are the sets of all vertices in $V(G) - S$ that are adjacent to none of S or to all of S, respectively, then A is an independent set and B is a clique. Hence $G = (G', A, B) \circ G[S]$, where $G' = G[A \cup B]$.*

Proof. If two vertices in A are non-adjacent or two vertices of B are adjacent, then these vertices are the midpoints of a (possibly closed) S-terminal alternating path of length 3, preventing S from being a strict module. □

Whenever $G = (G', A, B) \circ G_0$, the vertex set of G_0 is a strict module in G. We thus conclude the following.

Corollary 4.6. *A graph G is indecomposable with respect to canonical decomposition if and only if it has no nontrivial strict module.*

Corollary 4.6 shows that in the study of strict modules, indecomposable graphs play a role like that of prime graphs for modules, which we now recall. A graph is *prime* if it has no nontrivial modules. A module S is *proper* if $S \neq V(G)$; single-vertex modules are proper. Gallai [9] showed that if G and $\overline{G}$ are both connected, then every vertex in G belongs to a unique maximal proper module. The *modular decomposition tree* of a graph is obtained by recursively applying the following result.

Theorem 4.7 (Gallai [9]). *Let G be a graph with at least two vertices. Exactly one of the following conditions holds.*

(i) *G is disconnected.*

(ii) *$\overline{G}$ is disconnected.*

(iii) *The maximal proper modules partitition $V(G)$, and the subgraph induced by a set consisting of one vertex from each maximal proper module is a maximal prime subgraph.*

Jamison and Olariu [16] developed a refinement called *primeval decomposition* using the P_4-structure. A graph G is *p-connected* if for every partition of its vertex set into two nonempty disjoint sets, some edge in the P_4-structure intersects both sets. A maximal p-connected induced subgraph of G is a *p-component*. A p-connected graph G is *separable* if its vertex set splits into two nonempty disjoint sets such that each P_4 not contained within one of the sets has its endpoints in one set and its midpoints in the other. The primeval decomposition of a graph partitions its vertex set into modules via the following theorem.

Theorem 4.8 (Jamison–Olariu [16])**.** *For a graph G, exactly one of the following holds.*

(i) *G is disconnected.*

(ii) *$\overline{G}$ is disconnected.*

(iii) *G is p-connected.*

(iv) *There is a unique proper separable p-component Q of G with a partition Q_1, Q_2 of $V(Q)$ such that every vertex not in $V(Q)$ is adjacent to all of Q_1 and none of Q_2.*

We present in Theorem 4.9 an analogue of Theorem 4.8 for A_4-structure and canonical decomposition. Let an A_4-*component* of a graph G be a subgraph of G induced by the vertices of some component of the A_4-structure of G. Theorem 3.2 shows that the A_4-components of G are precisely the components of the canonical decomposition of G. Define a graph to be A_4-*separable* if its vertices can be partitioned into two nonempty sets V and W such that every 4-vertex induced subgraph having an alternating 4-cycle has an alternating 4-cycle whose vertices alternate between V and W, as illustrated in Figure 5.
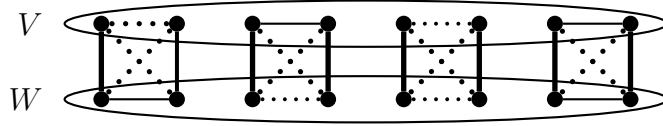

Figure 5: Alternating 4-cycles in an A_4-separable graph.

Split graphs with at least two vertices are A_4-separable. Each vertex of an alternating 4-cycle C has a neighbor and a nonneighbor along C, so no clique or independent set can have three vertices of C. Also, C cannot have consecutive vertices in a clique and the other two in an independent set, because adjacency in C is the same for both pairs. Hence every alternating cycle in a split graph alternates between the clique and the independent set.

Theorem 4.9. *For any graph G with more than one vertex having canonical decomposition (*) into $G_k, \ldots, G_0$, exactly one of the following is true:*

(i) *G has an isolated vertex.*

(ii) *$\overline{G}$ has an isolated vertex.*

(iii) *The A_4-structure of G is connected.*

(iv) *There is a unique proper A_4-separable A_4-component Q of G with a partition Q_1, Q_2 of $V(Q)$ such that every vertex not in $V(Q)$ is adjacent to none of Q_1 and to all of Q_2.*

Proof. Here "proper" means $V(Q) \neq V(G)$. Since $G_k, \ldots, G_0$ are the A_4-components of G, the only candidate for Q in (iv) is G_k; for any other G_i, the vertices of G_k ensure that no nontrivial partition of G_i can satisfy the adjacency condition.

Indecomposable graphs have no isolated or dominating vertex, so no two of (i), (ii), (iii) can simultaneously hold. If (iii) holds, then there is no proper A_4-component. If (i) or (ii) holds, then G_k is not A_4-separable. Hence at most one of the conditions holds.

If (iii) fails, then $k \geq 1$. If (i) and (ii) fail, then A_k and B_k are both nonempty. Since split graphs are A_4-separable, G_k is an A_4-component of G having the properties in (iv). We have noted that G_k is the only candidate, so at least one of the conditions holds. $\square$

5 A_4-split graphs

In this section we characterize the A_4-*split graphs*, those having the same A_4-structure as some split graph. As motivation, we show that this problem arises in the problem of constructing all graphs having a given A_4-structure.

Example 5.1. *Graphs with the same A_4-structures.* Proposition 3.11 states that every alternating 4-cycle in a graph G lies within one component of the canonical decomposition. Thus permuting the indecomposable components in a canonical decomposition does not change the A_4-structure $H(G)$.

By Theorem 3.2 and Proposition 3.11, each component of $H(G)$ is uniquely determined by the component of the canonical decomposition having the same vertex set. Replacing a component of the canonical decomposition with another subgraph having the same A_4-structure yields a graph with the same A_4-structure as G.

To illustrate these two operations, let G_2 be a copy of K_1 with vertex u, let G_1 be a copy of K_1 with vertex v, and let $G_0 = K_2 + P_3$. Given G with canonical decomposition $(G_2, \varnothing, \{u\}) \circ (G_1, \{v\}, \varnothing) \circ G_0$, let G' be the graph formed by transposing the first two components of the canonical decomposition; that is, $G' = (G_1, \{v\}, \varnothing) \circ (G_2, \varnothing, \{u\}) \circ G_0$. Let G'_0 be the 5-vertex graph with degree sequence $(3, 2, 1, 1, 1)$; note that G_0 and G'_0 have the same A_4-structure. Let G'' be the graph formed from G by replacing G_0 with G'_0; that is, $G'' = (G_2, \varnothing, \{u\}) \circ (G_1, \{v\}, \varnothing) \circ G'_0$. The graphs G, G', and G'' appear in Figure 6. Though the graphs are pairwise nonisomorphic, they have the same A_4-structure.
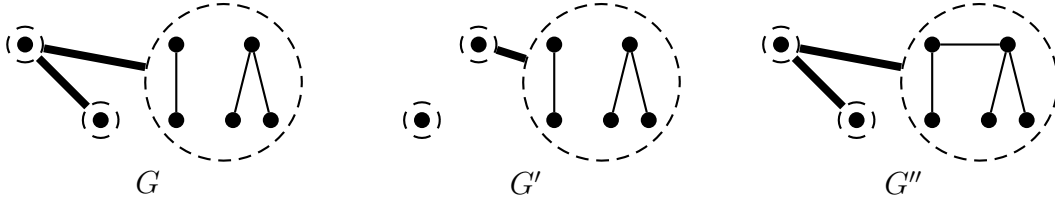

Figure 6: Different graphs with the same A_4-structure.

Given a graph G with canonical decomposition $(G_k, A_k, B_k) \circ \cdots \circ (G_1, A_1, B_1) \circ G_0$, call G_0 the *core* of G. The components of the decomposition other than the core are all split graphs. To generate other graphs having the same A_4-structure as G, we may wish to permute the indecomposable components of G under the canonical decomposition. If the core G_0 is not a split graph, then we cannot move the vertices of G_0 to a different position in the canonical decomposition unless we first replace G_0 by a split graph G'_0 having the same A_4-structure as G_0. To determine whether this is possible, we seek a characterization of the graphs having the same A_4-structure as a split graph; these are the A_4-*split graphs*. $\square$

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The characterization of A_4-split graphs uses several other concepts. A *split partition* of a split graph partitions the vertex set into a clique and an independent set. An *separating partition* of an A_4-separable graph G partitions $V(G)$ into two sets such that every induced subgraph having an alternating 4-cycle has an alternating 4-cycle that alternates between the two sets. A graph G and its A_4-structure are A_4-*balanced* if $V(G)$ can be partitioned into two sets such that every alternating 4-cycle of G has two vertices in each; the two sets then form a *balancing partition*. By definition, every A_4-separable graph is A_4-balanced; the converse fails, since every bipartite graph is A_4-balanced, but C_6 is not A_4-separable.

Given an A_4-balanced A_4-structure H with balancing partition $\{V_1, V_2\}$ and a vertex $v \in V_i$, the *v-restriction* of H is the graph on V_{3-i} in which two vertices are adjacent if and only if they both lie in some edge of H containing v. An A_4-balanced A_4-structure H has the *bipartite restriction property* if there is a balancing partition of H such that for all $v \in V(H)$ the v-restriction of H is bipartite.

The *k-pan* is the graph obtained by attaching a pendant vertex to a vertex of a k-cycle; the *co-k-pan* is its complement. The 4-pan and co-4-pan are shown in Figure 7.
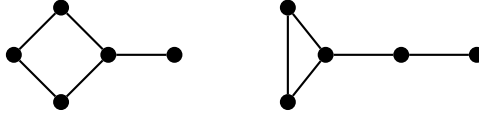

Figure 7: The 4-pan and the co-4-pan.

Theorem 5.2 (Földes–Hammer [6])**.** *G is a split graph if and only if G is $\{2K_2, C_4, C_5\}$-free.*

Corollary 5.3. *For any split partition (Q, S) of a split graph, every induced 4-vertex path has its midpoints in Q and its endpoints in S, and $\{Q, S\}$ is a balancing partition.*

Proof. The placement of 4-vertex paths is immediate. By Theorem 5.2, the vertices of every alternating 4-cycle induce P_4. □

Theorem 5.4. *For a graph G with core G_0 and A_4-structure H, the following statements are equivalent.*

(a) *G is A_4-split.*

(b) *H is A_4-balanced and has the bipartite restriction property.*

(c) *G and $\overline{G}$ are both $\{C_5, P_5, K_2 + K_3, co\text{-}4\text{-}pan, K_2 + P_4, K_2 + C_4, 2K_2 \vee 2K_1\}$-free.*

(d) *G is a split graph, or one of $\{G_0, \overline{G}_0\}$ is a disjoint union of stars.*

(e) *G is A_4-separable.*

Proof. We show that each condition implies the next and that (e) implies (a).

(a) $\Rightarrow$ (b). Let G' be a split graph with the same A_4-structure H as G, and let (Q, S) be a split partition of G'. By Corollary 5.3, $\{Q, S\}$ is a balancing partition, so H is A_4-balanced.

For any induced copy of P_4 in G' with vertices $a_1, a_2 \in Q$ and $b_1, b_2 \in S$, each a_i has one neighbor in $\{b_1, b_2\}$ and each b_i has one neighbor in $\{a_1, a_2\}$. Hence if $v \in V(G')$ and B is the v-restriction of H, then in G' vertex v has exactly one neighbor in each edge of B. Giving the neighbors and nonneighbors of v in $V(B)$ opposite colors thus yields a proper 2-coloring of B. Hence B is bipartite, and H has the bipartite restriction property.

(b) $\Rightarrow$ (c). Since a graph and its complement have the same A_4-structure, the existence of balancing partitions and the bipartite restriction property are preserved under complementation. They are also preserved under taking induced subgraphs. It thus suffices to show that the A_4-structure of each graph listed in (c) is not A_4-balanced or does not have the bipartite restriction property. For C_5, each 4-set contains an alternating 4-cycle, but no bipartition of $V(C_5)$ splits each 4-set equally. The co-4-pan, P_5, and $K_2 + K_3$ have the same A_4-structure H^* with three edges. In H^*, the only balancing partition has two vertices in one set and three vertices in the other. The v-restriction of H^* for a vertex v in the 2-set is K_3, so H^* does not have the bipartite restriction property. The A_4-structures of $K_2 + P_4$, $K_2 + C_4$, and $2K_2 \vee 2K_1$ also each have a unique balancing partition and a vertex v such that the v-restriction of the A_4-structure is K_3.

(c) $\Rightarrow$ (d). Suppose that neither G nor $\overline{G}$ has any graph listed in (c) as an induced subgraph. If G is not a split graph, then the core G_0 of G is not a split graph. Since by hypothesis G_0 is C_5-free, Theorem 5.2 implies that G_0 induces $2K_2$ or C_4. By complementation, we may assume that G_0 induces $2K_2$, with edges ab and cd; let $U = \{a, b, c, d\}$. Since G is $\{K_2 + K_3, P_5, \text{co-4-pan}\}$-free, every vertex of G_0 outside U has 0, 1, or 4 neighbors in U. Partition $V(G_0) - U$ into sets X, Y, A, B, C, and D, by those whose neighborhoods in U are U, $\varnothing$, $\{a\}$, $\{b\}$, $\{c\}$, and $\{d\}$, respectively (see Figure 8).
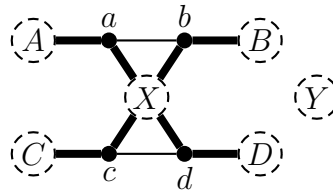

Figure 8: The graph G_0 from Theorem 5.4.

Since G_0 is $(2K_2 \vee 2K_1)$-free, X is a clique. If $X \neq \varnothing$, choose $x \in X$. Since G_0 is co-4-pan-free, $A = B = C = D = \varnothing$. Let Y'' be the set of isolated vertices in $G_0[Y]$. Let $Y' = Y - Y''$. Adjacent vertices $y_1, y_2 \in Y'$ are both adjacent to x; otherwise, $\{y_1, y_2, x, a, b\}$ induces $K_2 + K_3$ or the co-4-pan. Thus all of X is adjacent to all of Y'. Now $G_0 = (G_0[X \cup Y''], Y'', X) \circ G_0[\{a, b, c, d\} \cup Y']$, contradicting the indecomposability of G_0.

Hence $X = \varnothing$. Since G_0 is $(K_2 + P_4)$-free, A or B is empty, as is C or D. By symmetry, we may assume $B = D = \varnothing$. Since G_0 is $\{K_2 + K_3, P_5\}$-free, $A \cup C$ is independent. Since G_0 is $(K_2 + P_4)$-free, no vertex of Y has a neighbor in $A \cup C$. Thus $G_0[A \cup \{a, b\}]$ and $G_0[C \cup \{c, d\}]$ are components of G_0 that are stars. Since G_0 is $\{K_2 + K_3, K_2 + P_4, K_2 + C_4\}$-free, $G_0[Y]$ is $\{K_3, P_4, C_4\}$-free. The $\{K_3, P_4, C_4\}$-free graphs are forests with diameter at most 2 and hence also are disjoint unions of stars.

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(d) $\Rightarrow$ (e). If G is a split graph, then we have observed that a split partition of G is A_4-separating. Hence we may assume by complementation that G_0 is a disjoint union of stars. Let A' be a largest independent set in G_0, and let $B' = V(G_0) - A'$. Any 4-vertex induced subgraph of G_0 having an alternating 4-cycle is isomorphic to $2K_2$ and has two nonadjacent vertices in each of A' and B'; thus A', B' is A_4-separating. Since the other components $G_k, \ldots, G_1$ of the canonical decomposition are split graphs and every alternating 4-cycle lies entirely within a single component (by Proposition 3.11), the partition of $V(G)$ into sets $A_k \cup \cdots \cup A_1 \cup A'$ and $B_k \cup \cdots \cup B_1 \cup B'$ is A_4-separating.

(e) $\Rightarrow$ (a). Let V_1 and V_2 form an A_4-separating partition of $V(G)$. Obtain a split graph G' from G by deleting all edges of $G[V_1]$ and adding all edges missing from $G[V_2]$, so (V_2, V_1) is a split partition of G'. Let H' be the A_4-structure of G'; we claim that $H' = H$.

For each edge of H, there is an alternating 4-cycle in G that alternates between V_1 and V_2, and hence the cut between V_1 and V_2 on these four vertices is a matching. In G', these vertices induce P_4, so $E(H) \subseteq E(H')$. Conversely, in G' every alternating 4-cycle alternates between V_1 and V_2. Those edges and non-edges are the same as in G, so $E(H') \subseteq E(H)$. Thus equality holds, and G has the same A_4-structure as the split graph G'. $\square$

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