Research Statement
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My main research interests lie in structural and extremal graph theory, with secondary interests in combinatorics and linear algebra. One focus of my research is on understanding how limited or local information, which may accurately describe several different graphs, may be used to determine properties that hold for all graphs fitting that information. For instance, much of my work has dealt with the degree sequence of a graph, the list of integers recording how many edges are incident with each vertex in the graph. This is local information, as knowing how many edges meet at each vertex seems at first to give no indication of which vertices are adjacent to each other. It is also limited information in that most degree sequences belong to several nonisomorphic graphs (the *realizations* of the degree sequence). However, the degree sequence contains a surprising amount of information about a graph, and there is a rich interplay between the local degree information and both the local and the global structure of the graph. This structural information is necessarily true of *all* realizations of the degree sequence, which is helpful if the isomorphism class of the graph is not known. Information available from a degree sequence is also useful from a computational standpoint, as the degree sequence typically requires much less memory and time to store and manipulate than a complete specification of the graph does.

Another theme in my research is the analysis of graph structure in extremal problems on graphs. Extremal problems ask for the optimum values of graph parameters under various conditions. My research seeks to understand the configurations or properties present in a graph that force the parameter values to behave as they do, for such graph parameters as the reconstruction number, tree-depth, and independence number.

The following description of my work will focus in turn on my results on degree sequences and the structure they impose on realizations, hereditary families of graphs that are characterized by degree sequences, and structural results related to extremal problems.

Graphic sequences and their realizations

A list of nonnegative integers is *graphic* if it is the degree sequence of a simple graph. Not every list is graphic; there are many equivalent sets of conditions for deciding this for a given list (for a summary, see [29]). Comparing graphic and non-graphic lists of positive numbers, it is apparent that short lists are often not graphic. In joint work with S. G. Hartke, F. K. Jao, and D. B. West, I showed [11] that a partial converse holds for this statement: every list with an even sum that is "long enough" is graphic. We give a precise value for this length threshold in terms of the largest and smallest entries in the list and the maximum difference g between consecutive terms. Zverovich and Zverovich [40] found a threshold that is quadratic in the largest entry when no conditions on g are specified. Our work generalizes this result and shows that the threshold becomes linear, asymptotically, when g is fixed.

In [8] I introduced a relaxation of the notion of a realization of a degree sequence d by studying edge labelings of the complete graph with labels from the interval $[0,1]$ so that the sum of values on all edges meeting at a vertex equals a specified term of d. The labels in each of these "fractional" realizations are the coordinates of a point in $\mathbb{R}^{\binom{n}{2}}$, and the collection of all such points forms a convex polytope $P(d)$. Not surprisingly, the simple graph realizations of d correspond to vertices of $P(d)$. However, for many d there are non-integral vertices of $P(d)$ that do not correspond to simple graph realizations of d. I characterized the degree sequences d for which $P(d)$ has only integral vertices (the *decisive sequences*) in terms of configurations that cannot appear in any of their realizations and also in terms of the numerical values present in d. The class of these sequences properly includes the well-studied class of threshold sequences.

Given a graphic list, and considering now only simple graph realizations, what can the degree sequence guarantee about the structure of an arbitrary realization? In general, it may not be possible to conclude that two vertices with specified degrees will be adjacent (or non-adjacent), but in some cases it is. This is certainly true for the threshold graphs, those graphs in which *every* adjacency relationship is uniquely determined by the degree sequence. In [7] I determined necessary and sufficient conditions for a pair of vertices to be adjacent (or non-adjacent) in all of the realizations of a degree sequence, and I also determined the structure of the overall forced subgraphs present in a realization and its complement. I showed that in several ways these forced relationships give a measure of how close a degree sequence is to a threshold sequence.

In studying properties of the realizations of a degree sequence, it helps to understand how the realizations are linked. A well-known result in [24] states that two graphs have the same degree sequence if and only if one can be transformed into the other through a sequence of graph operations known as 2-switches. A 2-switch takes an *alternating 4-cycle*, a configuration on four vertices in which two edges and two non-edges alternate in a cyclic fashion, and toggles the status of the edges and non-edges. Understanding the changes possible through 2-switches offers insight into the differences among graphs with the same degree sequence. In [6] I studied conditions under which a 2-switch changes the isomorphism type of a graph. I showed that if a 2-switch results in a different isomorphism class, then the alternating 4-cycle involved must be a part of one of four larger configurations; if a graph contains neither of two additional configurations, then this necessary condition is also sufficient.

For graphs, 2-switches transform realizations of a degree sequence into each other. Similar ideas and operations apply to directed graphs, and in particular to tournaments (directed graphs where every pair of vertices is joined by exactly one directed edge). The tournament analogue of a degree sequence is a *score sequence*, the list of vertex outdegrees, and one analogue of a 2-switch involves reversing the arcs of directed triangles. In an attempt to find a closer digraph analogue to the notion of an alternating 4-cycle, D. Brown and I are currently studying the $\Box$-*interchange graph* $B(T)$ of a tournament T, in which vertices correspond to tournaments having a given score sequence and edges join tournaments linked by a single reversal of a directed 4-cycle [9]. The graph $B(T)$ is itself an analogue of the so-called realization graph of a graph degree sequence, which has been shown to be connected and sometimes Hamiltonian [2]. We have shown that for all but transitive tournaments, $B(T)$ always consists of two components, and under certain conditions it is a vertex-transitive graph.

Alternating 4-cycles play an important role in the study of degree sequences because of their presence in every 2-switch. In order to shed still further light on how the alternating 4-cycles of a graph interact with the degree sequence, in [16] D. West and I defined the A_4-*structure* of a graph G to be the 4-uniform hypergraph having the same vertex set as G in which four vertices form an edge if and only if they comprise the vertex set of an alternating 4-cycle in G. We showed that two graphs having the same A_4-structure are either both perfect or both imperfect, and if the graphs are also triangle-free, then the vertex sets of their matchings with at least two edges are identical in the two graphs. The A_4-structure also provides a new motivation for the *canonical decomposition* of a graph, as defined by Tyshkevich [36, 37]; in particular, a graph is canonically indecomposable if and only if its A_4-structure is connected. It follows that graphs with the same degree sequence agree on the numbers and sizes of connected components in their A_4-structures. I further showed in [6] that the Erdős–Gallai inequalities, classic conditions for recognizing degree sequences [22], actually provide information about the vertex sets of components in the decomposition or A_4-structure.

While the canonical decomposition is a useful tool for studying degree sequences, it has other uses as well, and sometimes these uses lead to unexpected degree sequence results. For instance, the Antimagic Labeling Conjecture states that any connected graph with m edges may have its edges labeled with $1, \ldots, m$ so that the sums at vertices are pairwise distinct [26]. In [3] I showed that the conjecture is true for split graphs and graphs with more than one component in the canonical decomposition. As a consequence, I derived a sufficient condition on degree sequences for antimagic labelings to exist for each of their realizations.

Hereditary classes characterized by degree sequences

A graph class is *hereditary* if whenever a graph is in the class, all induced subgraphs of the graph also belong to the class. Several fundamental classes of graphs are hereditary, including the bipartite graphs and the planar graphs. Given a hereditary class $\mathcal{G}$, every graph not belonging to $\mathcal{G}$ is a *forbidden subgraph for* $\mathcal{G}$, since that graph does not appear as an induced subgraph of any element of $\mathcal{G}$. Knowing the forbidden subgraphs for $\mathcal{G}$ that are minimal under the induced subgraph partial order is one way to characterize graphs in $\mathcal{G}$.

A number of well known hereditary graph classes have degree sequence characterizations, that is, it is possible to determine whether a graph belongs to the class knowing only its degree sequence. This requires that for every degree sequence d, either all or none of the realizations of d belong to the class. Examples of such classes include the families of threshold graphs, split graphs, and matrogenic graphs. Each of these classes is known for strict structure imposed on its graphs and for easily stated forbidden subgraph characterizations (see [29]).

As outlined below, my work has defined and characterized new hereditary families having degree sequence characterizations, and has provided context for all such families.

In [8] I introduced the class of *decisive graphs*. These graphs are the realizations of the decisive sequences described above in connection with the polytope of fractional realizations. I characterized these graphs in terms of 70 forbidden induced subgraphs and also in terms of their canonical decompositions; a graph is decisive if and only if any non-split canonical component has one of a small number of forms. The decisive graphs properly include all threshold, split, and pseudo-split graphs.

In a series of two papers [5, 6] I studied hereditary classes of unigraphs. A *unigraph* is a graph that is the unique realization, up to isomorphism, of its degree sequence. In [6] my study of 2-switches provided a new characterization of the matrogenic graphs, a well-studied hereditary class of unigraphs. I also defined the *hereditary unigraphs* to be the unigraphs having only unigraphs as induced subgraphs. The hereditary unigraphs comprise the largest hereditary class containing only unigraphs, and as such they properly contain the threshold graphs, matroidal graphs, and matrogenic graphs. In [6] I characterized the hereditary unigraphs in terms of forbidden configurations and forbidden induced subgraphs. In [5] I further characterized the hereditary unigraphs in terms of their canonical decompositions and degree sequences, and I showed how these characterizations naturally generalize those known for threshold, matroidal, and matrogenic graphs.

In addition to studying specific hereditary families with degree sequence characterizations, I have worked to develop a theory of all such classes. Together with M. Kumbhat and S. Hartke, I have studied the sets of minimal forbidden induced subgraphs for these families. In [12] we defined a set $\mathcal{F}$ of graphs to be *degree-sequence-forcing (DSF)* if the class of $\mathcal{F}$-free graphs has a degree sequence characterization. We showed that every DSF set must contain a disjoint union of complete graphs, a complete multipartite graph, a forest of stars, and the complement of a forest of stars. We characterized the DSF sets of size at most 2 and the *non-minimal* DSF triples, i.e., DSF sets of three graphs with a proper DSF subset [12, 14]. Hartke and I continued this work in [13] by designing a computer algorithm that identifies all minimal DSF triples. In preparation for the algorithm, we established further conditions on DSF sets and on minimal DSF sets in particular. Of note, we showed that there are finitely many minimal DSF sets of any given size, and we presented a bound on the orders of the graphs these sets contain.

Structural aspects of extremal problems

Broadly speaking, extremal graph theory studies the optimization of graph parameters. Often graphs that optimize a given parameter have an elegant structure that allows them to do so. Part of my work has involved determining various parameters for graphs and studying how these values are related to aspects of the graphs' structure.

In ongoing work [10], M. Ferrara, J. Vandenbussche, P. Wenger, and I examine a colored version of a classical graph saturation parameter. A t-edge-colored graph is a simple graph in which each edge has been labeled with a value (a "color") from $\{1, \dots, t\}$. A rainbow copy of H is obtained from H by coloring its edges so that no two edges have the same color. We define $\mathrm{sat}_t(H, n)$ to be the minimum number of edges in an n-vertex, t-edge-colored graph G that contains no rainbow copy of H but has the property that adding any edge to G, in any color, creates a rainbow copy of H. We show that this parameter is well defined for all t and H. We provide various upper and lower bounds on $\mathrm{sat}_t(H, n)$ when H is a path, cycle, star, matching, or complete graph, as well as when H satisfies various other structural properties.

In [4] I compared the independence number $\alpha(G)$, i.e., the maximum number of pairwise non-adjacent vertices that can be found in G, to the residue $r(G)$, a parameter that is computed through an iterative reduction process on the degree sequence of G. It is known [23] that the residue is always a lower bound for the independence number of a graph. However, $r(G)$ is computed from the degree sequence of G, which may also be the degree sequence of several other graphs, and these graphs may vary widely on their independence numbers, so $r(G)$ and $\alpha(G)$ may differ greatly; indeed, in [4] I showed that the difference may be arbitrarily large. However, it is an open question as to how much $r(G)$ may differ from $\alpha(G)$ when G has the lowest independence number among all the realizations of d. Nelson and Radcliffe [30] showed that if d is a graphic list whose maximum and minimum values differ by at most 1, then d has a realization G such that $\alpha(G)$ and $r(G)$ may differ by at most 1. Using the canonical decomposition and a characterization of the unigraphs by Tyshkevich [37], I showed that the same is true of all unigraphs; in fact, I precisely determined the graphs for which the difference is 1. Along the way I eliminated most of the need for iteration in computing the residue by reducing the problem to finding the residue of the canonical "core" of a graph, the only non-iterative result to date that simplifies computation of $r(G)$.

One of the most famous open problems in graph theory is the Graph Reconstruction Conjecture [27, 38]. It states that each graph on at least 3 vertices is uniquely determined by its *deck*, the multiset of induced subgraphs (called *cards*) obtained by deleting one vertex from the graph. Results so far have shown how to determine many of a graph's properties from its deck, and the conjecture has been proved for various classes of graphs. However, the general problem remains open at this time. A parameter introduced by S. Ramachandran for measuring how hard it is to reconstruct a graph is *degree-associated reconstruction number* $\mathrm{drn}(G)$ of a graph G. This parameter was defined in [32] as the minimum number of degree-associated cards (i.e., pairs consisting of a card and the degree in G of the deleted vertex) that suffice to uniquely determine G. The Reconstruction Conjecture is equivalent to showing that $\mathrm{drn}(G)$ is defined for each graph G. In [17] Douglas B. West and I observed that $\mathrm{drn}(G) \leq 2$ for almost all graphs G (asymptotically), and we characterized the graphs for which $\mathrm{drn}(G) = 1$. We obtained $\mathrm{drn}(G)$ for all G in various graph classes. In particular, it is known that $\mathrm{drn}(T) \leq 3$ for any tree T; we showed that for all caterpillars but one, $\mathrm{drn}(T) \leq 2$, and we conjecture that there are at most finitely many trees with $\mathrm{drn}(T) = 3$. We also showed that $\mathrm{drn}(G) \geq 3$ for all vertex-transitive graphs other than complete graphs, and that equality holds for hypercubes. In a sense, this says that vertex-transitive graphs are generally harder to reconstruct. Future work may extend our results on vertex-transitive graphs. Understanding better the difficulties in reconstructing these graphs may lead to better understanding of how to reconstruct any graph.

The *tree-depth* (also known as the vertex ranking number) of a connected graph G is the minimum integer k such that it is possible to label the vertices of G with values from $\{1, \ldots, k\}$ so that every path joining vertices with the same label also contains a vertex with a higher label. The tree-depth is minor-monotonic, so if G contains H as a minor, then the tree-depth of G is at least as large as the tree-depth of H. In [15] J. Sinkovic and I studied minor-minimal graphs having a desired tree-depth k; these are *k-critical graphs*. We showed that several families of graphs were k-critical, and we gave an inductive construction that produces a large class $\mathcal{M}_k$ of k-critical graphs. Our construction relies on a property we called *1-uniqueness*, and we conjectured that all critical graphs are 1-unique. If this is the case, then $\mathcal{M}_k$ provides a framework in which to understand *all* critical graphs with tree-depth k.

Future directions

Following are some research questions that I plan to pursue in the near future.

Which hereditary graph families have degree sequence characterizations? This question has motivated my research on degree-sequence-forcing sets, and as mentioned above, my coauthors and I have placed a number of hereditary families from the literature in context by examining small sets of forbidden subgraphs. However, classical hereditary families such as the bipartite graphs, the planar graphs, and the perfect graphs are alike in that they have no degree sequence characterization and cannot be characterized by a finite number of forbidden induced subgraphs. This leads me to conjecture that *every hereditary family with a degree sequence characterization must have a finite number of minimal forbidden induced subgraphs*, and I expect that my research in the near future will focus in part on proving this conjecture. As a beginning, S. Hartke and I have introduced the set $\mathcal{D}(G)$ for a graph G, the set of all minimal graphs (under the induced subgraph order) having the same degree sequence as a graph that induces G [13]. We characterized DSF sets in terms of the $\mathcal{D}$-sets of their elements, and we found nearly all the minimal DSF triples by explicitly finding the eight graphs in $\mathcal{D}(K_3)$. In general, I conjecture that $\mathcal{D}(G)$ contains finitely many graphs, and I have a construction that I believe generates the largest graphs possible for $\mathcal{D}(G)$. The truth of this latter conjecture, in conjunction with a recent finiteness result of Chudnovsky and Seymour [20], would imply that every hereditary family with a degree sequence has a characterization in terms of finitely many forbidden subgraphs.

My work on hereditary families with degree sequence characterizations has focused largely on the sets of minimal graphs that *could not* appear as induced subgraphs of graphs in the family. Turning the question around, several unsolved questions concern the *presence*, rather than absence, of a particular induced subgraph or subgraphs. Given a graph G, is it true that there is always a degree sequence for which every realization contains G as an induced subgraph? Results in this direction could lead to better bounds in terms of the degree sequence on such graph parameters as the matching, clique, and independence numbers, improving upon results such as those of Caro and Wei [19, 39] and Favaron et al. [23]. It seems likely that a resolution to such "forced subgraph" problems will require approaches quite different from those used to study forbidden subgraphs. As first steps towards a theory in this area, we might characterize the graphs that can be induced in a unigraph (since being induced in all realizations of a degree sequence is trivially satisfied); in one sense this problem serves as a complement to the problem of characterizing the hereditary unigraphs [5, 6]. We might also characterize the subgraphs we can build using the forced edges and forced non-adjacencies described in [7]; such subgraphs must clearly appear in every realization of the degree sequence used.

Finally, having attended the Rocky Mountain Mathematics Consortium Summer School on algebraic graph theory in June 2013, and at present collaborating on a grant proposal to organize a conference on Ramanujan graphs, I look forward to expanding my research into algebraic aspects of graphs. One question I plan to investigate is under what circumstances membership in a hereditary family may be recognized by examining a graph's eigenvalues. This is a spectral analogue of the motivating question behind degree-sequence-forcing sets, and, given that triangle-free graphs and bipartite graphs are large graph classes that have spectral characterizations of this type, the question promises to have an interesting answer.

References

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